MXB261: Week 2 Workshop - Maths Refresher I & II

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How to use this worksheet

This worksheet covers a range of maths you will need for this unit. It is deliberately longer than one workshop — around five hours of work in total — so you are *not* expected to finish it today. Use it as a practice resource and checklist. During the workshop, prioritise questions where the answer sheet won't be enough, where a tutor can help you most.

Contents

1 Sets and probability

Filter form:

$$E = \left\{ \overbrace{x}^{\text{dummy variable}} \in \overbrace{\mathbb{Z}^+}^{\text{generator}} \underbrace{\mid}_{\text{such that}} \underbrace{x \text{ is even and } x < 10}_{\text{guard, condition}} \right\}$$

Map form:

$$E = \left\{ \overbrace{2x}^{\text{output expression}} \underbrace{\mid}_{\text{such that}} x \in \overbrace{\mathbb{Z}^+}^{\text{generator}}, \underbrace{x < 5}_{\text{guard, condition}} \right\}$$

Mnemonic: $\in$lement

Subset:

$$A \subseteq B \quad \text{means} \quad x \in A \implies x \in B$$

Proper subset:

$$A \subsetneq B \quad \text{means} \quad A \subseteq B \text{ but } A \neq B$$

Mnemonics: $\subseteq$ is like $\leq$, $\subsetneq$ is like $<$

Intersection:

$$A \cap B = \{x \mid x \in A \text{ and } x \in B\}$$

Mnemonic: i$\cap$tersection, a$\cap$d

Union:

$$A \cup B = \{x \mid x \in A \text{ or } x \in B\}$$

Mnemonic: $\cup$nion, $\cup$r

Set difference:

$$A \setminus B = \{x \mid x \in A \text{ and } x \notin B\}$$

Set complement:

$$A^c = \{x \mid x \in \Omega \text{ and } x \notin A\}$$

where Ω is the universal set.

Question 1.

1. Let $A = \{1, 3, 5, 7, 9\}$. Fill in $\in$ or $\notin$:

$$3 \text{ ...... } A, \qquad 4 \text{ ...... } A, \qquad 9 \text{ ...... } A$$

2. Let $B = \{2, 4, 6, 8, 10\}$. Is the statement $7 \notin B$ true or false?
3. List the elements of
$$C = \{x \in \mathbb{Z}^+ \mid x \text{ is odd and } x < 8\}.$$
Hint: $\mathbb{Z}^+$ means positive integers, i.e., $1, 2, 3, \ldots$.
4. List the elements of
$$D = \{x \in \mathbb{Z}^+ \mid x^2 < 20\}.$$
5. Find the elements of
$$G = \{x \in \mathbb{Z}^+ \mid x < 1\}.$$
6. List the elements of
$$E = \{2n + 1 \mid n \in \mathbb{Z}^+ \text{ and } n \leq 4\}.$$
7. Write the set $\{3, 6, 9, 12\}$ using set-builder notation.
8. Write the set $\{2, 4, 6, 8, 10\}$ using set-builder notation.
9. Write the set $\{1, 4, 9, 16, 25\}$ using set-builder notation.
10. True or false: $\{x \in \mathbb{Z}^+ \mid x \text{ is even and } x \text{ is odd}\} = \emptyset$.
11. Let $A = \{1, 2, 3, 4, 5\}$ and $B = \{2, 4, 6, 8\}$. Find $A \cap B$.
12. Let $P = \{1, 3, 5\}$ and $Q = \{2, 4, 6\}$. Find $P \cap Q$.
13. Let $A = \{1, 2, 3\}$ and $B = \{3, 4, 5\}$. Find $A \cup B$.
14. Let $M = \{2, 4, 6\}$ and $N = \{1, 2, 3, 4, 5, 6\}$. Find $M \cup N$.
15. Let $A = \{x \in \mathbb{Z}^+ \mid x < 6\}$ and $B = \{x \in \mathbb{Z}^+ \mid x \text{ is even and } x < 10\}$. Find $A \cap B$ and $A \cup B$.

Union:

$$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$

Disjoint intersection:

$$A \cap B = \emptyset \quad \Longrightarrow \quad P(A \cap B) = 0$$

Independent intersection (special case):

$$P(A \cap B) = P(A) \cdot P(B)$$

Dependent intersection (general case):

$$P(A \cap B) = P(A \mid B) \cdot P(B)$$

Complement:

$$P(A^c) = 1 - P(A)$$

Question 2. Consider a standard deck of 52 cards with no Jokers. We assume the deck is well shuffled and we pick a single card at random.

1. What is the sample space (universal set)?
2. By the Principle of Indifference, what is the probability of picking a $7\heartsuit$?
3. What is the probability of picking a 7?
4. What is the probability of picking a $\heartsuit$?
5. What is the probability of picking a card that is not a $\heartsuit$?
6. What is the probability of picking a card that is a 7 or a $\heartsuit$?

Question 3. A season of 100 soccer matches is recorded. For each match we note whether the home team won, and whether the match was decided by a margin of at least two goals.

	Margin ≥ 2 (B)	Margin < 2	Total
Home team won (A)	25	21	46
Home team did not win	14	40	54
Total	39	61	100

Let A be the event "the home team won" and B the event "the margin was at least two goals".

1. Find $P(A)$, $P(B)$ and $P(A \cap B)$.
2. Are A and B independent? Justify your answer.
3. Find $P(A \mid B)$ and interpret it in words.
4. Find $P(A \cup B)$.
5. Let E be the event "the match was a draw" Explain why A and E are disjoint. What is $P(A \cap E)$?

2 Calculus refresher

2.1 Calculus rules cheat sheet

Differentiation rules

- Power Rule: $\frac{d}{dx}[x^n] = nx^{n-1}$
- Constant Multiple Rule: $\frac{d}{dx}[cf(x)] = c\frac{d}{dx}[f(x)]$
- Sum/Difference Rule: $\frac{d}{dx}[f(x) \pm g(x)] = \frac{d}{dx}[f(x)] \pm \frac{d}{dx}[g(x)]$
- Product Rule: $\frac{d}{dx}[u(x)v(x)] = u'(x)v(x) + u(x)v'(x)$
- Quotient Rule: $\frac{d}{dx}\left[\frac{u(x)}{v(x)}\right] = \frac{u'(x)v(x) - u(x)v'(x)}{[v(x)]^2}$
- Chain Rule: $\frac{d}{dx}[f(g(x))] = f'(g(x)) \cdot g'(x)$

Common derivatives

- $\frac{d}{dx}[c] = 0$ (constant)
- $\frac{d}{dx}[e^x] = e^x$
- $\frac{d}{dx}[e^{ax}] = ae^{ax}$
- $\frac{d}{dx}[\ln(x)] = \frac{1}{x}$
- $\frac{d}{dx}[\sin(x)] = \cos(x)$
- $\frac{d}{dx}[\cos(x)] = -\sin(x)$
- $\frac{d}{dx}[\tan(x)] = \sec^2(x)$

Integration Rules

- Power Rule: $\int x^n \, dx = \frac{x^{n+1}}{n+1} + C \quad$ (where $n \neq -1$)
- Constant Multiple Rule: $\int cf(x) \, dx = c \int f(x) \, dx$
- Sum/Difference Rule: $\int [f(x) \pm g(x)] \, dx = \int f(x) \, dx \pm \int g(x) \, dx$

- Fundamental Theorem of Calculus: $\int_a^b f(x)\,dx = F(b) - F(a)$

  where $F(x)$ is an antiderivative of $f(x)$

Common integrals

- $\int k\,dx = kx + C$ (constant)
- $\int \frac{1}{x}\,dx = \ln|x| + C$
- $\int e^x\,dx = e^x + C$
- $\int e^{ax}\,dx = \frac{1}{a}e^{ax} + C$
- $\int \sin(x)\,dx = -\cos(x) + C$
- $\int \cos(x)\,dx = \sin(x) + C$
- $\int \sin(ax)\,dx = -\frac{1}{a}\cos(ax) + C$
- $\int \cos(ax)\,dx = \frac{1}{a}\sin(ax) + C$

Note: C represents the constant of integration for indefinite integrals.

2.2 Practice questions

Question 4. Evaluate the following derivatives:

1. $\frac{d}{dx}(1+x)(x^2+2)$
2. $\frac{d}{dx}x^2\ln(3x)$
3. $\frac{d}{dt}\left(\frac{e^{-t}}{t^2}\right)$

Question 5. Evaluate the following indefinite integrals.

1. $\int 5x^4\,dx$
2. $\int \frac{1}{x}\,dx$

Question 6. Evaluate the following definite integrals:

1. $\displaystyle\int_{-1}^{1} \cos(2\pi t)\ dt$
2. $\displaystyle\int_{0}^{\pi} e^{2x}\ dx$

3 Maxima and minima

3.1 Twice-differentiable functions and local maxima and minima

In the lecture, I found and classified the stationary points of the curve

$$y = f(x) = x^4 - 4x^3 + 5$$

To find the stationary points, I found the values of x for which the first derivative equaled zero. I found the first derivatives:

$$f'(x) = 4x^3 - 12x^2 = 4x^2(x-3)$$

And evaluating at zero:

$$f'(x) = 4x^2(x-3) = 0 \implies x^* = 0 \text{ or } 3$$

Substituting each x^* into $y = x^4 - 4x^3 + 5$ gives stationary points: (0, 5) and (3, -22).

To classify the stationary points, I first tried the second-derivative test. I found the second derivative:

$$f''(x) = 12x^2 - 24x = 12x(x-2)$$

For $(x^*, y^*) = (3, -22)$:

$$f''(3) = 36 > 0;\ \text{a minimum}$$

For $(x^*, y^*) = (0, 5)$:

$$f''(x^* = 0) = 0;\ \text{ambiguous}$$

To resolve the ambiguous case $x^* = 0$, I chose $a < x^* < b$ close enough that no stationary point lies between a and x^* nor x^* and b.

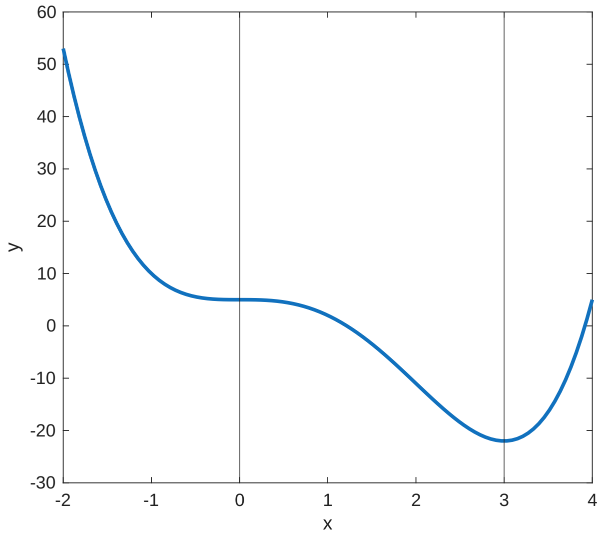
Choosing $-1 < 0 < 1$:

$$f'(-1) < 0 \text{ and } f'(1) < 0;\ \text{inflection point (falling)}$$

To check my answer, I can use the following plotting script:

```
clearvars
% define function, bounds, crit pts
x_lo = -2; x_hi = 4;
f = @(x) x.^4 - 4 * x.^3 + 5;
crit_xs = [0, 3];

% evaluate and plot fnc, crit pts
x = linspace(x_lo, x_hi, 100);
y = f(x);
plot(x, y, "LineWidth", 2);
for k = 1:length(crit_xs)
  xline(crit_xs(k));
end
xlabel("x"); ylabel("y")
```



Question 7. Find the stationary points of the following functions and classify them.

1. $f(x) = x^3 - 6x^2 + 9x + 1$
2. $f(x) = x^2 e^{-x}$
3. $f(x) = x^5 - 5x^4 + 5$

Hint: The plotting script above has been written so you can update just the first block to check your answers.

3.2 Global optima

Recall that when finding optima on a bounded function, one must also check the endpoints.

Question 8. Find the global optima (global maximum and minimum) of the following differentiable functions

1. $f(x) = x^3 - 3x$ on $[-2, 3]$
2. $f(x) = x^3 - 3x$ on $[-\frac{3}{2}, \frac{3}{2}]$ (Note: Same f as previous question, different endpoints)
3. $f(x) = x^2 e^{-x}$ on $[-1, 3]$ (Note: Same f as Q 7.2)

Recall that if the infimum (or supremum) of a function is not attained, then the function has no global minimum (or maximum).

Question 9. Find the global optima of

$$f(x) = x \qquad \text{on } [0, 1).$$

In the lectures, I discussed the piecewise function

$$f(x) = \begin{cases} (x-1)^2 & 0 \le x < 3 \\ |x-4| + 1 & 3 \le x \le 6 \end{cases} \quad \text{on } [0, 6]$$

To find the global maximum and minimum (if they existed), I first identified suspect points: stationary points (i.e., points where $f'(x) = 0$), endpoints of the domain, interior points where f is undefined or jumps, and continuous points where f' doesn't exist.

I tabulated each point c below, and I calculated the limit of the function approaching from the left

$$f(c^-) := \lim_{x \to c^-} f(x)$$

from the right

$$f(c^+) := \lim_{x \to c^+} f(x)$$

and at the point $f(c)$.

c	Type	$f(c^-)$	$f(c)$	$f(c^+)$
0	endpoint	—	1	1
$\mathbf{1}$	stationary point ($f' = 0$)	0	$\mathbf{0}$	0
$\mathbf{3}$	seam	$\mathbf{4}$	2	2
4	corner (f' doesn't exist)	1	1	1
6	endpoint	3	3	—

I found the infimum by scanning for the lowest value in the table, which was $\inf f = 0$. It appeared in the $f(c)$ column and therefore was attained. Therefore

Global minimum: $(1, 0)$

I found the supremum by scanning for the highest value in the table, which was $\sup f = 4$. It did not appear in the $f(c)$ column and therefore was not attained. Therefore

No global maximum

Here is the script I used to plot the function in MATLAB. You can copy and paste it and update the `pieces` array to check your answers to the questions below.

```
clear all
% specify each piece here
pieces = {
%   formula             a   a incl?     b   b incl?
    @(x) (x-1).^2,      0,  true,       3,  false
    @(x) abs(x-4) + 1,  3,  true,       6,  true
};

col = [0 0.45 0.74];            % colour for curves and markers (a blue)
hold on                         % keep every piece on the same axes

for k = 1:size(pieces, 1)
    f     = pieces{k, 1};       % the formula for this piece
    a     = pieces{k, 2};       % left end
    aIncl = pieces{k, 3};       % is the left end included?
    b     = pieces{k, 4};       % right end
    bIncl = pieces{k, 5};       % is the right end included?

    % draw the curve for this piece
    xx = linspace(a, b, 400);            % 400 points across the piece
    yy = arrayfun(f, xx);                % evaluate the formula at each one
    plot(xx, yy, 'LineWidth', 2, 'Color', col);

    % draw a circle at each end
    ends = {a, aIncl; b, bIncl};         % the two ends of this piece
    for j = 1:2
        xe   = ends{j, 1}; incl = ends{j, 2}; ye   = f(xe);
        if incl
            faceColour = col;            % filled  -> included / attained
        else
            faceColour = 'w';            % open    -> not attained
        end
        plot(xe, ye, 'o', 'MarkerSize', 8, ...
             'MarkerFaceColor', faceColour, ...
             'MarkerEdgeColor', col, 'LineWidth', 1.5);
    end
end
```

```
grid on
xlabel('x')
ylabel('f(x)')
title('Piecewise function')
hold off
```

Question 10. Find the global optima of

$$f(x) = \begin{cases} x & 0 \leq x \leq 2 \\ 3 - x & 2 < x < 4 \end{cases} \qquad \text{on } [0, 4).$$

Question 11. Find the global optima of

$$f(x) = \begin{cases} -x^2 + 6x & 0 \leq x < 2 \\ 10 - 2x & 2 \leq x \leq 4 \end{cases} \qquad \text{on } [0, 4].$$

Question 12. Find the global optima of

$$f(x) = \begin{cases} -\frac{1}{4}(x-1)^2 + 5 & 1 \leq x < 3 \\ x + 1 & 3 \leq x \leq 5 \end{cases} \qquad \text{on } [1, 5].$$

4 Linear algebra + linear programming

Question 13. Perform the following matrix multiplications by hand and check your answers using MATLAB. If the multiplication is not possible, explain why.

1. $$\begin{bmatrix} 1 & 2 \end{bmatrix} \begin{bmatrix} 0 & 4 \\ 2 & -1 \end{bmatrix}$$
2. $$\begin{bmatrix} 1 \\ 2 \end{bmatrix} \begin{bmatrix} 0 & 4 \\ 2 & -1 \end{bmatrix}$$
3. $$\begin{bmatrix} -1 & 5 \\ 2 & 3 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 3 & 1 \end{bmatrix}$$
4. $$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ -3 & 0 \end{bmatrix}$$
5. $$\begin{bmatrix} 5 & 3 \end{bmatrix} \begin{bmatrix} -1 \\ 2 \end{bmatrix}$$

4.1 Solving systems of linear equations

Question 14. Consider the following system of equations

$$\begin{aligned} 4x + 3y &= -1 \\ 8x + 7y &= -5 \end{aligned}$$

1. Write the system in matrix form
2. Solve the system using Gaussian Elimination or simultaneous equations.
3. Check your answer using MATLAB (Hint: `inv(A)`)

Question 15. When you try to use MATLAB to solve the following system of equations

$$
\begin{aligned}
x + 3y + 5z &= 4 \\
x + 2y - 3z &= 5 \\
2x + 5y + 2z &= 8
\end{aligned}
$$

it returns a warning and an answer with infinity and `NaN` values. Why?

Hint: The solution to a 3×3 system is the intersection of three planes.

4.2 Optimisation using linear programming

In the lectures, I solved the following optimisation problem:

$$
\begin{array}{rcccccc}
\text{maximize:} & 30w & + & 40c & & \\
\text{subject to:} & w & + & 2c & \leq & 24 \\
 & w & + & c & \leq & 16 \\
 & 3w & + & 2c & \leq & 44
\end{array}
$$

I wrote the maximisation problem as a minimisation problem in matrix form:

$$
\text{minimise: } \underbrace{\begin{bmatrix} -30 & -40 \end{bmatrix}}_{\mathbf{f}^T} \underbrace{\begin{bmatrix} w \\ c \end{bmatrix}}_{\mathbf{x}}
$$

$$
\text{subject to: } \underbrace{\begin{bmatrix} 1 & 2 \\ 1 & 1 \\ 3 & 2 \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} w \\ c \end{bmatrix}}_{\mathbf{x}} \leq \underbrace{\begin{bmatrix} 24 \\ 16 \\ 44 \end{bmatrix}}_{b}
$$

Then I solved it in MATLAB using `linprog`:

```
>> f = [-30; -40];
>> A = [1 2; 1 1; 3 2];
>> b = [24; 16; 44];
>> x = linprog(f, A, b)
Optimal solution found.
x =
     8
     8
```

Question 16. Farmers earn \$1.2/kg of profit for potatoes and \$1.7/kg for carrots. A farmer has enough seed to plant 3 tonnes of potatoes, 4 tonnes of carrots, and enough fertiliser for 5 tonnes of crops. Use `linprog` in MATLAB to calculate how many kilograms of potatoes and carrots she should plant to maximise her profits. How much profit will she earn?

4.3 Finding eigenvalues and eigenvectors

Question 17. Consider the matrix

$$A = \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix}$$

With pen and paper, I found the following eigenvalue and eigenvector pairs:

$$\lambda_1 = 1, \quad \mathbf{v}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$\lambda_2 = 3, \quad \mathbf{v}_2 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

Use the MATLAB function `eig` to check my answer.

Hint: Remember that the eigenvectors indicate a direction, so if $\mathbf{v}$ is an eigenvector, then so is $k\mathbf{v}$ for any scalar k.

5 Probability distributions

5.1 Continuous probability distribution

Question 18. Consider the probability density function

$$f(x) = 3x^2 \text{ on } [0, 1]$$

1. Verify that $f(x)$ is a valid probability density function.
2. Calculate its expected value.
3. Calculate its variance.
4. Write its cumulative distribution function.
5. Use the cumulative distribution function to write an anonymous function `inv_cdf(u)` that draws random values from $f(x)$ using the inverse transform sampling method.

6. Verify that your function `inv_cdf(u)` is sampling correctly by plotting $f(x)$ on top of a histogram of 1000 randomly sampled x values. You can use the normalisation style `'pdf'` so the histogram scales correctly.

```
histogram(x_samples, "Normalization", "pdf");
```

Hint: See example in Lecture Week 2.

5.2 Discrete probability distribution

Question 19. You buy a $80 concert ticket with plans to resell it for a profit. There are 3 possible outcomes:

1. The show blows up and you resell for $400 (probability .25)
2. The show does okay, you resell at face value $80 (probability .35)
3. The show flops, nobody wants the ticket, you get $0 (probability .40)

You are interested in investigating the profit you could make from this scheme.

Write a script that calculates and displays the following:

1. Expected profit
2. Variance
3. Standard deviation
4. Probability of not making positive profit

Hint: See example in Lecture Week 2.

6 Space cows (Markov chain)

Imagine that we have been following a herd of cows that we suspect have been trampling on some precious native habitat. The cows move between a field, a waterhole, and the native habitat, but very rarely stay in the native habitat for more than a day at a time, meaning rangers are unsure about whether the cows are a serious problem. Through some state of the art research done by the CSIRO "Space Cows" program (a real program), we have been able to gather enough information to partially fill out a transition matrix for how the cows move between locations on a day to day basis.

Question 20. Write down the state space of the Markov Chain problem. How many states are there?

Let us index the locations 1 field, 2 waterhole, and 3 native habitat. We can write a transition matrix T where T_{ij} is the probability of moving from location j to location i. Fill out the missing entries of the transition matrix below so that it meets the requirements to be a proper transition matrix.

$$T = \begin{bmatrix} 0.5 & 0.2 & 0.35 \\ 0.1 & 0.4 & \ldots \\ \ldots & \ldots & 0.1 \end{bmatrix}$$

Question 21. Write a script that simulates the location of a cow for 100 timesteps, starting at location 1 (field), and stores the location in an array `locations`.

Create a figure with two subplots: the cow's location through time, and a frequency histogram of the cow's location. You can use the following template:

```
% assume some code above produces the array `locations` that records the cow's
% location at each timestep, e.g., locations = [1; 3; 3; 2; ...]

% subplot(1, 2, 1) means a figure with 1 x 2 panels
% and plot the following in panel 1
subplot(1, 2, 1);
plot(...)
xlabel("timestep")
ylabel("location")
title("space cow trajectory")

subplot(1, 2, 2); % plot the following in panel 2
histogram(locations(2:end), "Normalization", "probability")
xlabel("location")
ylabel("frequency")
title("cow location frequencies")
```

Hints

How do I...?	**See**
write and run a script?	Week 1 Sect. 1.6
simulate over 100 timesteps?	`for` loop, Week 1 Sect. 1.8
determine the cow's new location at each timestep?	see tree-stump game Lect. 2
store each new location in the `locations` array?	Week 1 Sect. 1.10

Question 22. Use the `eig` function (see Q. 17) to find the eigenvalues and eigenvectors of the space cows transition matrix. In the long term, what proportion of time will the cows spend in the field, waterhole, and native habitat?

7 Answers

Sets and probability

Question 1:

1. $3 \in A, \quad 4 \notin A, \quad 9 \in A.$
2. True, since 7 is not one of the listed elements of B.
3. $C = \{1, 3, 5, 7\}$.
4. $D = \{1, 2, 3, 4\}$, since $1, 4, 9, 16 < 20$ but $25 \not< 20$.
5. $G = \emptyset$, since no positive integer is less than 1.
6. $E = \{3, 5, 7, 9\}$, from substituting $n = 1, 2, 3, 4$ into $2n + 1$.
7. One correct form is
$$\{x \in \mathbb{Z}^+ \mid x \text{ is a multiple of } 3 \text{ and } x \leq 12\}.$$
8. One correct form is
$$\{x \in \mathbb{Z}^+ \mid x \text{ is even and } x \leq 10\}.$$
9. One correct form is
$$\{x^2 \mid x \in \mathbb{Z}^+ \text{ and } x \leq 5\},$$
i.e. the squares of the first five positive integers.
10. True. No integer is both even and odd, so the set has no elements.
11. $A \cap B = \{2, 4\}$.
12. $P \cap Q = \emptyset$ (the sets are disjoint).
13. $A \cup B = \{1, 2, 3, 4, 5\}$. Note that 3 is listed only once.
14. $M \cup N = \{1, 2, 3, 4, 5, 6\} = N$, since $M \subseteq N$.
15. $A = \{1, 2, 3, 4, 5\}$ and $B = \{2, 4, 6, 8\}$, so
$$A \cap B = \{2, 4\}, \qquad A \cup B = \{1, 2, 3, 4, 5, 6, 8\}.$$

Question 2:

1. A sample space is the set of all possible outcomes of an experiment:

$$\begin{aligned}\Omega = \{&A\heartsuit, 2\heartsuit, \ldots, 10\heartsuit, J\heartsuit, Q\heartsuit, K\heartsuit, && \text{(13 }\heartsuit\text{'s)}\\ &A\diamondsuit, 2\diamondsuit, \ldots, 10\diamondsuit, J\diamondsuit, Q\diamondsuit, K\diamondsuit, && \text{(13 }\diamondsuit\text{'s)}\\ &A\spadesuit, 2\spadesuit, \ldots, 10\spadesuit, J\spadesuit, Q\spadesuit, K\spadesuit, && \text{(13 }\spadesuit\text{'s)}\\ &A\clubsuit, 2\clubsuit, \ldots, 10\clubsuit, J\clubsuit, Q\clubsuit, K\clubsuit\} && \text{(13 }\clubsuit\text{'s)}\\ & && \text{(52 cards total)}\end{aligned}$$

2. By the Principle of Indifference, each card is equally likely. Therefore, dividing the total probability 1 by the size of the sample space, the probability of picking a $7\heartsuit$ is 1/52.

3. Write A for the event "the card is a 7"

$$A = \{7\heartsuit, 7\diamondsuit, 7\spadesuit, 7\clubsuit\} \qquad \text{(4 outcomes)}$$

   Counting outcomes of interest and dividing by the size of the sample space

$$P(A) = \frac{4}{52} = \frac{1}{13}$$

4. Write B for the event "the card is a heart"

$$B = \{A\heartsuit, 2\heartsuit, \ldots, 10\heartsuit, J\heartsuit, Q\heartsuit, K\heartsuit\} \qquad \text{(13 outcomes)}$$

   Counting outcomes of interest and dividing by the size of the sample space:

$$P(B) = \frac{13}{52} = \frac{1}{4}$$

5. If B is the set of hearts, then B^c is the set of not-hearts.

$$P(B^c) = 1 - P(B) = 1 - \frac{1}{4} = \frac{3}{4}.$$

6. There are four 7s and thirteen hearts, but we must be careful not to simply add these together because that would count the $7\heartsuit$ card twice.

   The union rule avoids the double counting:

$$P(A \cup B) = P(A) + P(B) - P(A \cap B) = \frac{4}{52} + \frac{13}{52} - \frac{1}{52} = \frac{16}{52} = \frac{4}{13}$$

   Alternatively, we can again count outcomes of interest and divide by the size of the sample space. The outcomes of interest are

$$\begin{aligned} A \cup B = \{A\heartsuit, 2\heartsuit, \ldots, 7\heartsuit, \ldots, K\heartsuit, & \qquad \text{(13 cards)} \\ 7\diamondsuit, 7\spadesuit, 7\clubsuit\} & \qquad \text{(3 cards)} \\ & \qquad \text{(16 cards total)} \end{aligned}$$

So

$$P(A \cup B) = 16 \times \frac{1}{52} = \frac{4}{13}$$

Question 3:

1. Reading the totals straight off the table,

$$P(A) = \frac{46}{100} = 0.46, \qquad P(B) = \frac{39}{100} = 0.39, \qquad P(A \cap B) = \frac{25}{100} = 0.25$$

   The intersection is the single cell where both conditions hold: 25 matches were home wins *and* decided by two or more goals.

2. If the events were independent we would expect

$$P(A) \cdot P(B) = 0.46 \times 0.39 = 0.1794$$

   But we observed $P(A \cap B) = 0.25$, which is noticeably larger. Since $P(A \cap B) \neq P(A) \cdot P(B)$, the events are not independent. This may be due to the 'home team advantage' from the lecture.

3. For dependent events, we must use the conditional probability:

$$P(A \mid B) = \frac{P(A \cap B)}{P(B)} = \frac{0.25}{0.39} \approx 0.641$$

   Given that a match was decided by at least two goals, there is roughly a 64% chance the winner was the home team, which is larger than $P(A) = 0.46$. So learning that the match was one-sided makes a home win substantially more likely.

   Equivalently, we can read this off the table directly: of the 39 matches with a margin of at least two goals, 25 were won by the home team, and $25/39 \approx 0.641$.

4. Using the union rule and being careful to subtract the overlap:

$$P(A \cup B) = P(A) + P(B) - P(A \cap B) = 0.46 + 0.39 - 0.25 = 0.60$$

   So 60 of the 100 matches were either a home win, a two-goal-or-more margin, or both.

5. A match cannot both be a draw and a home win, $P(A \cap E) = 0$.

Calculus refresher

Question 4:

1. $\dfrac{d}{dx}(1+x)(x^2+2)$

   We have the product of two terms, so we refer to the product rule from our cheat sheet

   > Product rule:
   > $$\frac{d(uv)}{dx} = u'v + v'u$$

   We will call the first term u and the second v

   $$\frac{d}{dx}\underbrace{(1+x)}_{u}\underbrace{(x^2+2)}_{v}$$

   $u' = 1$ and $v' = 2x$, so

   $$\begin{aligned}\frac{d}{dx}(1+x)(x^2+2) &= 1(x^2+2) + 2x(1+x)\\ &= x^2 + 2 + 2x + 2x^2\\ &= 3x^2 + 2x + 2\end{aligned}$$

2. $\dfrac{d}{dx}x^2\ln(3x)$

   In addition to the product rule, the following two rules from our cheat sheet look handy

   > $$\frac{d}{dx}\ln(x) = \frac{1}{x}$$

   > Chain rule:
   > $$\frac{d}{dx}f(g(x)) = f'(g)g'(x)$$

   First, we apply the product rule

   $$\frac{d}{dx}\underbrace{x^2}_{u}\underbrace{\ln(3x)}_{v}$$

   We have $u' = 2x$, but we're not sure about v'.

We apply the chain rule to get v'. Define

$$v(g(x)) = \ln(g) \text{ where } g(x) = 3x$$

Then using the chain rule

$$v'(g(x)) = v'(g)g'(x)$$

For $v(g) = \ln(g)$, $v'(g) = 1/g$.

For $g(x) = 3x$, $g'(x) = 3$.

Substituting in

$$v'(g(x)) = \frac{3}{g} = \frac{3}{3x} = \frac{1}{x}$$

Now we have our v', we return to the product rule we started out with

$$\begin{aligned} \frac{d}{dx}x^2 \ln(3x) &= u'v + v'u \\ &= 2x\ln(3x) + \frac{1}{x} \cdot x^2 \\ &= 2x\ln(3x) + x \end{aligned}$$

3. $\frac{d}{dt}\left(\frac{e^{-t}}{t^2}\right)$

We have the quotient of two terms, so we refer to the quotient rule from our cheat sheet

Quotient rule:

$$\frac{d}{dx}\left(\frac{u}{v}\right) = \frac{u'v - v'u}{v^2}$$

For $v = t^2$, $v' = 2t$, but for $u = e^{-t}$, we're not sure about u'.

To get u', we can apply the chain rule above. Another equivalent way to write the chain rule, which you might find easier to memorise, is

Chain rule:

$$\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx}$$

Define

$$u(t) = e^{-t} = e^{g(t)} \text{ where } g(t) = -t$$

Then

$$\frac{du}{dg} = e^g \text{ and } \frac{dg}{dt} = -1$$

Therefore

$$\begin{aligned}\frac{du}{dt} &= \frac{du}{dg}\frac{dg}{dt} \\ &= -e^{g} \\ &= -e^{-t}\end{aligned}$$

Now we have our u', we can return to using the quotient rule above

$$\begin{aligned}\frac{d}{dt}\frac{e^{-t}}{t^2} &= \frac{u'v - v'u}{v^2} \\ &= \frac{-e^{-t}t^2 - 2te^{-t}}{t^4} \\ &= \frac{-e^{-t}(t^2 + 2t)}{t^4} \\ &= -e^{-t}\left(\frac{1}{t^2} + \frac{2}{t^3}\right)\end{aligned}$$

Question 5:

1. $\displaystyle\int 5x^4 \; dx$

   There are two rules in our cheat sheet that can help us:

   Constant multiple rule

   $$\int cf(x) \; dx = c\int f(x) \; dx$$

   Power rule

   $$\int x^n \; dx = \frac{x^{n+1}}{n+1} + c \text{ for } n \neq -1$$

   Applying the two rules in turn

   $$\begin{aligned}\int 5x^4 \; dx &= 5\int x^4 \; dx \\ &= 5\left(\frac{x^5}{5} + c\right) \\ &= x^5 + 5c \\ &= x^5 + C \quad (5c = C \text{ is just another constant of integration})\end{aligned}$$

2. $\displaystyle\int \frac{1}{x} \; dx$

   At first, we might try to use our power rule

$$\int \frac{1}{x}\,dx = \int x^{-1}\,dx$$

However, we notice that the power is -1, and the power rule cannot be used when the power is -1, so we consult our cheat sheet again.

We find that the integral we're being asked to evaluate has its own rule

$$\int \frac{1}{x}\,dx = \ln|x| + c$$

Question 6:

1. $\displaystyle\int_{-1}^{1} \cos(2\pi t)\,dt$

   We find the following rule in our cheat sheet

   $$\int \cos(ax)\,dx = \frac{1}{a}\sin(ax) + c$$

   We have $a = 2\pi$, and we can use it directly

   $$\begin{aligned}\int_{-1}^{1} \cos(2\pi t)\,dt &= \left[\frac{1}{2\pi}\sin(2\pi t) + c\right]_{-1}^{1}\\ &= \left(\frac{1}{2\pi}\sin(2\pi) + c\right) - \left(\frac{1}{2\pi}\sin(-2\pi) + c\right)\\ &= \frac{1}{2\pi}\underbrace{\sin(2\pi)}_{0} - \frac{1}{2\pi}\underbrace{\sin(-2\pi)}_{0}\\ &= 0\end{aligned}$$

2. $\displaystyle\int_{0}^{\pi} e^{2x}\,dx$

   We find the following rule in our cheat sheet

   $$\int e^{ax}\,dx = \frac{1}{a}e^{ax} + c$$

   We have $a = 2$ and we can use it directly

$$\begin{aligned}\int_0^{\pi} e^{2x}\,dx &= \left[\frac{e^{2x}}{2}+c\right]_0^{\pi}\\ &= \frac{e^{2\pi}}{2}-\frac{e^0}{2}\\ &= \frac{1}{2}\left(e^{2\pi}-1\right)\end{aligned}$$

Maxima and minima

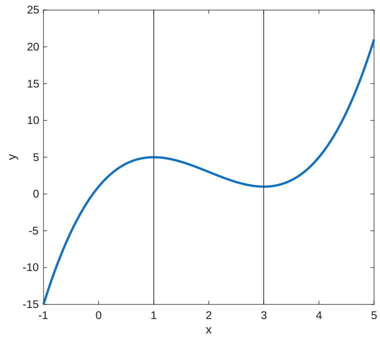
Question 7:

1. $f'(x) = 3x^2 - 12x + 9 = 3(x-1)(x-3) \implies x^* = 1,\ 3.$

   $f''(x) = 6x - 12.$

   $f''(1) = -6 < 0 \implies$ local max at $(1, 5)$.

   $f''(3) = +6 > 0 \implies$ local min at $(3, 1)$.

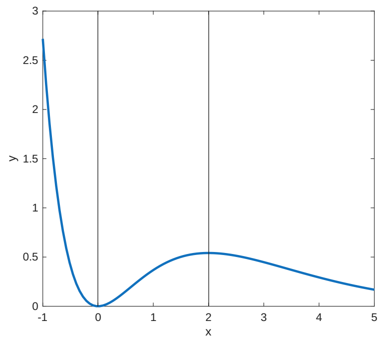


2. $f'(x) = 2xe^{-x} - x^2e^{-x} = x(2-x)e^{-x} \implies x^* = 0,\ 2$ (since $e^{-x} \neq 0$).

   $f''(x) = e^{-x}\left(x^2 - 4x + 2\right).$

   $f''(0) = 2 > 0 \implies$ local min at $(0, 0)$.

   $f''(2) = e^{-2}(4-8+2) = -2e^{-2} < 0 \implies$ local max at $(2,\ 4e^{-2}) \approx (2,\ 0.541)$.



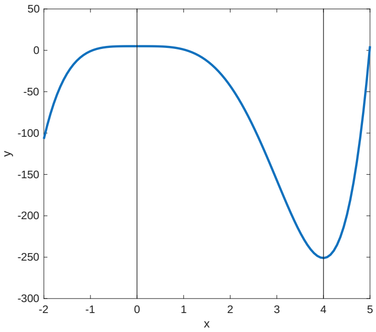
3. $f'(x) = 5x^4 - 20x^3 = 5x^3(x-4) \implies x^* = 0,\ 4.$

   $f''(x) = 20x^3 - 60x^2 = 20x^2(x-3).$

$f''(4) = 320 > 0 \implies$ local min at $(4, -251)$.

$f''(0) = 0$ inconclusive. First-derivative check: near $x = 0$, $(x - 4) < 0$, so f' has the sign of $-x^3$: positive for $x < 0$, negative for $x > 0$. Sign change from $+$ to $- \implies$ local max at $(0, 5)$.

So at $x = 0$ the graph has a local max even though the curve is momentarily flat to second order.



Question 8:

1. $f'(x) = 3x^2 - 3 = 3(x-1)(x+1) \implies x^* = -1,\ 1$, both interior.

   $f''(x) = 6x$:

   $f''(-1) = -6 < 0 \implies$ local max at $x = -1$,

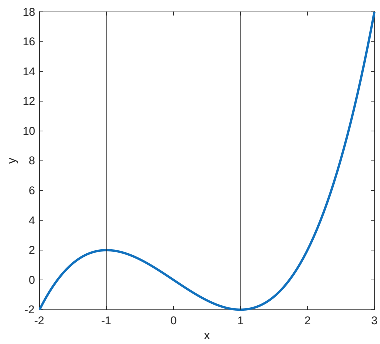
   $f''(1) = 6 > 0 \implies$ local min at $x = 1$.

   Tabulate every candidate:

x	$f(x)$
-2 (endpoint)	-2
-1 (local max)	2
1 (local min)	-2
3 (endpoint)	18

   Global maximum: $f(3) = 18$, at the right endpoint.

   Global minimum: -2, attained twice: at the interior local min $x = 1$ and at the endpoint $x = -2$.



2. As in the previous question, $f'(x) = 3x^2 - 3 = 3(x-1)(x+1) \implies x^* = -1,\ 1$, both interior.

   $f''(x) = 6x$:

   $f''(-1) = -6 < 0 \implies$ local max at $x = -1$,

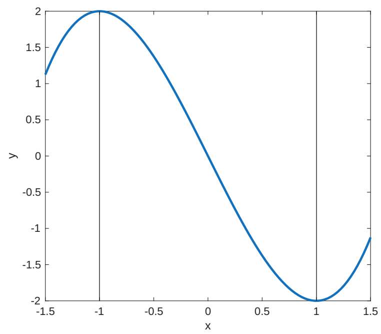
   $f''(1) = 6 > 0 \implies$ local min at $x = 1$.

   Tabulate every candidate:

x	$f(x)$	approx.
$-\frac{3}{2}$ (endpoint)	$\frac{9}{8}$	1.125
-1 (local max)	2	2
1 (local min)	-2	-2
$\frac{3}{2}$ (endpoint)	$-\frac{9}{8}$	-1.125

   Global maximum: $f(-1) = 2$, at the interior local max.

   Global minimum: $f(1) = -2$, at the interior local min.

3. $f'(x) = 2xe^{-x} - x^2e^{-x} = x(2-x)e^{-x} \implies x^* = 0,\ 2$ (as $e^{-x} \neq 0$).

   $f''(x) = e^{-x}(x^2 - 4x + 2)$:

   $f''(0) = 2 > 0 \implies$ local min at $x = 0$,

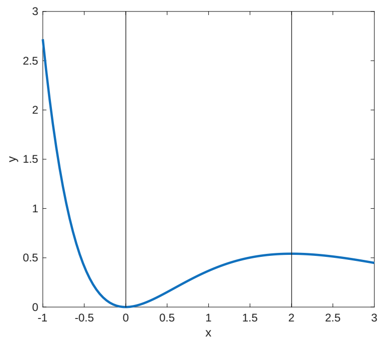
   $f''(2) = -2e^{-2} < 0 \implies$ local max at $x = 2$.

   Tabulate every candidate:

x	$f(x)$	approx.
-1 (endpoint)	e	2.718
0 (local min)	0	0
2 (local max)	$4e^{-2}$	0.541
3 (endpoint)	$9e^{-3}$	0.448

   Global maximum: $f(-1) = e \approx 2.718$, at the left endpoint.

   Global minimum: $f(0) = 0$, at the interior stationary point.



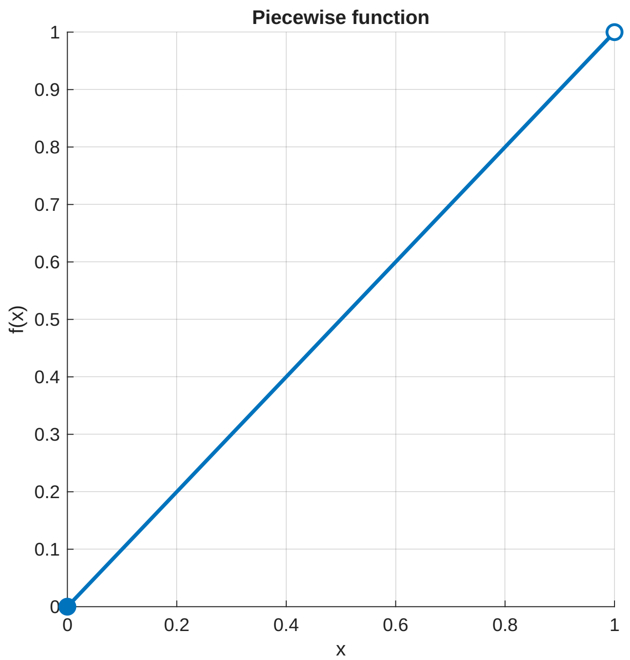
Question 9:

There are no stationary points ($f' = 1 \neq 0$) and no seams. The function is a single straight line, continuous and smooth everywhere on its domain. The only special points are the boundary ends: the included left end $x = 0$, and the *excluded* right end, approached as $x \to 1^-$.

Point c	Type	$f(c^-)$	$f(c)$	$f(c^+)$
$x = 0$	endpoint	—	0	0
$x = 1$	endpoint	1	—	—

$\inf f = 0$. Attained. $\implies$ global minimum (0, 0).

$\sup f = 1$. Not attained. $\implies$ no global maximum.



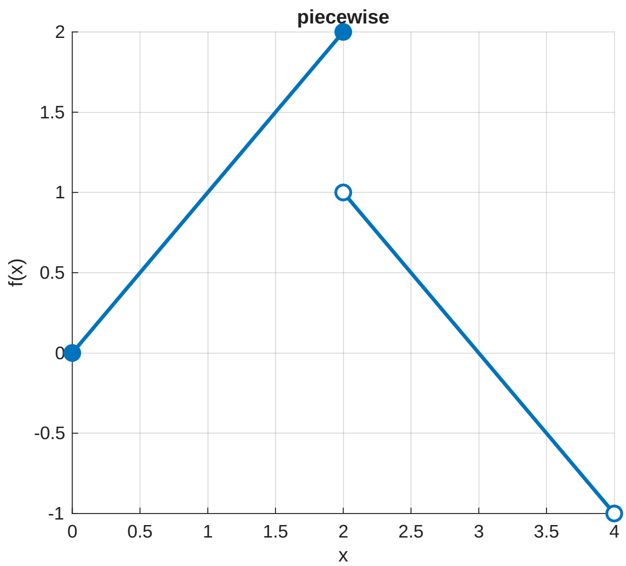
Question 10:

No stationary points: $f' = 1$ on the left piece and $f' = -1$ on the right. The only candidates are the endpoints $x = 0$ and $x = 4$ and the seam $x = 2$.

Point c	Type	$f(c^-)$	$f(c)$	$f(c^+)$
$x = 0$	endpoint	—	0	0
$x = 2$	seam	2	2	1
$x = 4$	endpoint	-1	-	—

$\inf f = -1$. Not attained. $\implies$ no global minimum.

$\sup f = 2$. Attained. $\implies$ global maximum (2, 2).



Question 11:

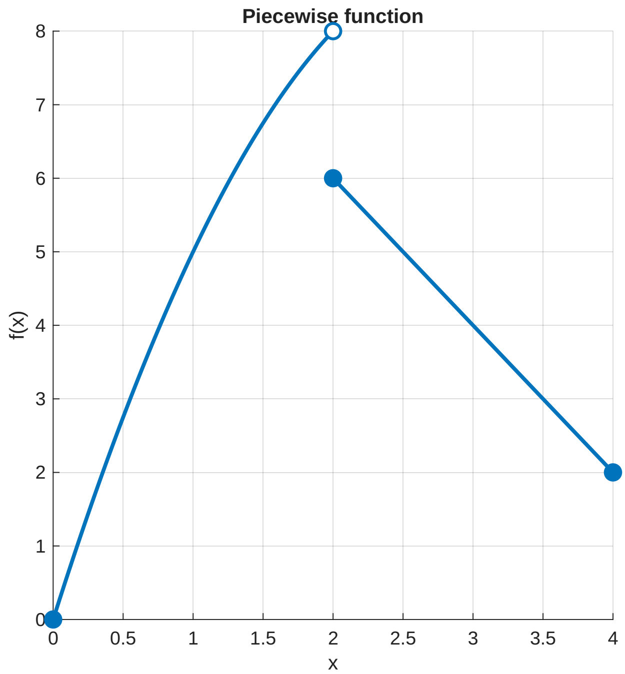
On the left piece, $f'(x) = -2x + 6 = 0$ gives a stationary point at $x = 3$ but $x = 3$ lies outside $[0, 2)$, so it is not a candidate.

On the right piece, $f' = -2 \neq 0$, so no stationary point there. The remaining candidates are the endpoints $x = 0$ and $x = 4$ and the seam $x = 2$.

Point c	Type	$f(c^-)$	$f(c)$	$f(c^+)$
$x = 0$	endpoint	—	0	0
$x = 2$	seam	8	6	6
$x = 4$	endpoint	2	2	—

$\inf f = 0$. Attained. $\implies$ global minimum (0, 0).

$\sup f = 8$. Not attained. $\implies$ no global maximum.



Question 12:

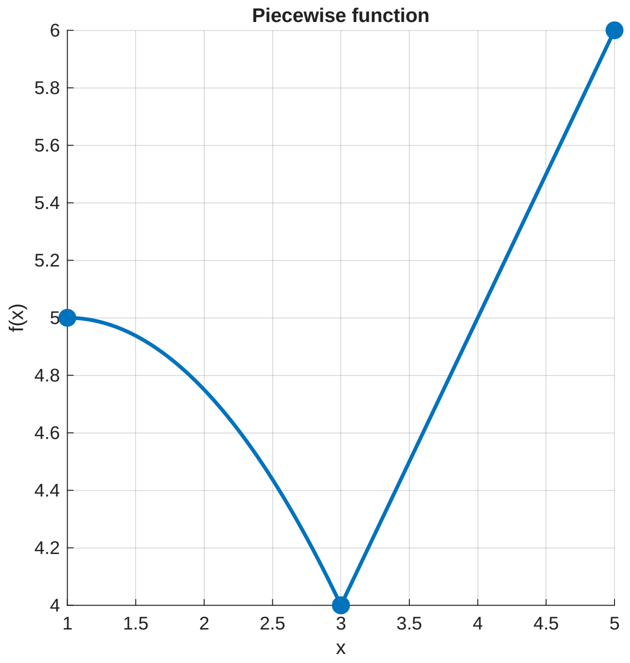
On the left piece, $f'(x) = -\frac{1}{2}(x-1) = 0$ gives a stationary point at $x = 1$ at the left endpoint of its range. Since $f''(x) = -\frac{1}{2} < 0$, it is a local maximum, value $f(1) = 5$. On the right piece $f' = 1 \neq 0$, there is no stationary point, and the piece is increasing. Candidates: the stationary point $x = 1$, the endpoint $x = 5$, and the seam $x = 3$.

Point c	Type	$f(c^-)$	$f(c)$	$f(c^+)$
$x = 1$	endpoint / stationary	—	5	5
$x = 3$	seam	4	4	4
$x = 5$	endpoint	6	6	—

Every candidate value is attained (no jumps nor unreached limits), so the global optima are simply the largest and smallest entries.

$\inf f = 4$. Attained. $\implies$ global minimum (3, 4).

$\sup f = 6$. Attained. $\implies$ global maximum (5, 6).



Linear algebra + linear programming

Question 13:

To check your answers, write your matrices in MATLAB using spaces (or commas) to separate columns and semicolons to separate rows (see Week 1 Workshop Sect. 1.2). Then enter

```
>> A = [1 2]; B = [0 4; 2 -1];
>> A * B
```

If you receive the error below

```
Error using *
Incorrect dimensions for matrix multiplication. Check that the number of columns in
    the first matrix matches the number of rows in the second matrix. To operate on
    each element of the matrix individually, use
TIMES (.*) for elementwise multiplication.
```

that means your matrix dimensions were incompatible. Two matrices can only by multiplied together if the number of columns in the first matrix equals the number of rows in the second.

1. $\begin{bmatrix} 1 & 2 \end{bmatrix} \begin{bmatrix} 0 & 4 \\ 2 & -1 \end{bmatrix} = \begin{bmatrix} 4 & 2 \end{bmatrix}$
2. Not possible: $(2 \times 1)(2 \times 2)$ inner dimensions $1 \neq 2$.
3. $\begin{bmatrix} -1 & 5 \\ 2 & 3 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 3 & 1 \end{bmatrix} = \begin{bmatrix} 14 & 3 \\ 11 & 7 \end{bmatrix}$

4. $I \begin{bmatrix} 1 & 1 \\ -3 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ -3 & 0 \end{bmatrix}$ multiplying by the identity matrix leaves the matrix unchanged

5. $\begin{bmatrix} 5 & 3 \end{bmatrix} \begin{bmatrix} -1 \\ 2 \end{bmatrix} = 1$

Question 14:

1. The system can be written in matrix form as $A\mathbf{x} = \mathbf{b}$:

$$\begin{bmatrix} 4 & 3 \\ 8 & 7 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -1 \\ -5 \end{bmatrix}$$

2. We solve the system by row-reducing the augmented matrix to echelon form.

$$\left[\begin{array}{cc|c} 4 & 3 & -1 \\ 8 & 7 & -5 \end{array}\right]$$

Eliminate the entry below the first pivot using $R_2 \to R_2 - 2R_1$:

$$\left[\begin{array}{cc|c} 4 & 3 & -1 \\ 8-2(4) & 7-2(3) & -5-2(-1) \end{array}\right] = \left[\begin{array}{cc|c} 4 & 3 & -1 \\ 0 & 1 & -3 \end{array}\right]$$

The augmented matrix is now in echelon form, and corresponds to the equations

$$\begin{aligned} 4x + 3y &= -1 \\ y &= -3 \end{aligned}$$

The second row gives y directly:

$$y = -3$$

Substitute $y = -3$ into the first equation and solve for x:

$$\begin{aligned} 4x + 3(-3) &= -1 \\ 4x - 9 &= -1 \\ 4x &= 8 \\ x &= 2 \end{aligned}$$

Therefore the solution is

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 2 \\ -3 \end{bmatrix}$$

3. To solve in MATLAB

```
>> A = [4 3; 8 7]
A =
     4     3
     8     7

>> b = [-1; -5]
b =
    -1
    -5

>> A \ b        % or inv(A) * b
ans =
     2
    -3
```

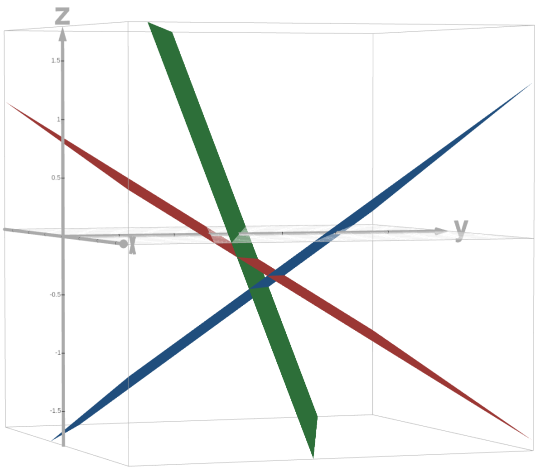
If you would like to learn more about Gaussian Elimination as an alternative to simultaneous equations, you may enjoy this video: https://www.youtube.com/watch?v=2tlwSqblrvU.

Question 15:

Your MATLAB output will look like:

```
>> A = [1 3 5; 1 2 -3; 2 5 2];
>> b = [4; 5; 8];
>> A \ b
Warning: Matrix is singular to working precision.
ans =
   NaN
  -Inf
   Inf
```

The reason why is because the matrix A is singular. In particular, this linear system has no solution. Recall that one interpretation of the system is that we are looking for the intersection of the three planes defined by the three equations. However, when we plot those planes (below), we see that they do not intersect at a single point.



Click to view a live version: https://www.desmos.com/3d/5qoyongrxm

Question 16:

Define:

$$\begin{aligned} p &= \text{kgs of potatoes planted} \\ c &= \text{kgs of carrots planted} \end{aligned}$$

Write problem:

$$\begin{array}{rrcrcrl} \text{maximize:} & 1.2p & + & 1.7c & & & \text{(maximise profits)} \\ \text{subject to:} & p & & & \leq & 3000 & \text{(seed available for potatoes)} \\ & & & c & \leq & 4000 & \text{(seed available for carrots)} \\ & p & + & c & \leq & 5000 & \text{(fertiliser available for both crops)} \end{array}$$

MATLAB's `linprog` function needs the problem specified in the following way

$$\begin{aligned} \text{minimise:} \quad & \mathbf{f}^T\mathbf{x} \\ \text{subject to:} \quad & A\mathbf{x} \leq \mathbf{b} \end{aligned}$$

and will return a solution for $\mathbf{x}$.

First, define a vector for the variables to solve $\mathbf{x}$

$$\mathbf{x} = \begin{bmatrix} p \\ c \end{bmatrix}$$

Next, turn our maximisation problem into a minimisation problem:

$$\text{maximize: } 1.2p + 1.7c \quad \longrightarrow \quad \text{minimise: } -1.2p - 1.7c$$

Write the minimisation problem in matrix form:

$$\text{minimise } -1.2p - 1.7c \quad \rightarrow \quad \text{minimise } \underbrace{\begin{bmatrix} -1.2 & -1.7 \end{bmatrix}}_{\mathbf{f}^T} \underbrace{\begin{bmatrix} p \\ c \end{bmatrix}}_{\mathbf{x}}$$

And write the constraints in matrix form:

$$\text{subject to: } \underbrace{\begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 1 \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} p \\ c \end{bmatrix}}_{\mathbf{x}} \leq \underbrace{\begin{bmatrix} 3000 \\ 4000 \\ 5000 \end{bmatrix}}_{b}$$

In MATLAB, at the prompt or in a script:

```
>> f = [-1.2; -1.7];
>> A = [1 0; 0 1; 1 1];
>> b = [3000; 4000; 5000];
>> x = linprog(f, A, b)
Optimal solution found.
x =
        1000
        4000

>> -f' * x
ans =
        8000
```

So the farmer should plant 1 tonne of potatoes and 4 tonnes of carrots, and she will earn $8000 of profit.

If you are new to linear programming and would like to learn more, you may enjoy this YouTube video: https://www.youtube.com/watch?v=E72DWgKP_1Y It goes through the problem above in more detail.

Question 17:

To use the MATLAB function `eig` to find the eigenvalues and eigenvectors:

```
>> A = [2 -1; -1 2];
>> [V, D] = eig(A)   % V: eigendirections columnwise, D: eigenvalues on diagonal
```

```
V =
   -0.7071   -0.7071
   -0.7071    0.7071

D =
     1     0
     0     3
```

`D` is a matrix with the eigenvalues on the diagonal, $\lambda_1 = 1$ and $\lambda_2 = 3$. `V` is a matrix with the corresponding eigenvectors written columnwise. Note that each column of `V` can be scaled to obtain the eigenvectors found with pen and paper, i.e., `V(:,1) / V(2,1)` and `V(:,2) / V(2,2)`. Therefore, the corresponding eigenvectors can also be written: $\mathbf{v}_1 = (1, 1)$ and $\mathbf{v}_2 = (-1, 1)$.

Probability distributions

Question 18:

1. To verify that $f(x)$ is a valid probability density function, we must verify that the total probability equals 1, i.e., $\int_0^1 f(x)\,dx = 1$.

$$\int_0^1 f(x)\,dx = \int_0^1 3x^2\,dx = \left[x^3\right]_0^1 = 1$$

2. The expected value

$$E(X) = \int_0^1 x f(x)\,dx = \int_0^1 3x^3\,dx = \left[\frac{3}{4}x^4\right]_0^1 = \frac{3}{4}$$

3. The variance

$$\mathsf{Var}(X) = \int_0^1 x^2 f(x)\,dx - E(X)^2$$

   The left-hand integral

$$\int_0^1 x^2 f(x)\,dx = \int_0^1 3x^4\,dx = \left[\frac{3x^5}{5}\right]_0^1 = \frac{3}{5}$$

   Therefore, the variance

$$\mathsf{Var}(X) = \frac{3}{5} - \left(\frac{3}{4}\right)^2 = \frac{48-45}{80} = \frac{3}{80}$$

4. The cumulative distribution function

$$F(t) = P(X \le t) = \int_0^t f(x)\,dx = \int_0^t 3x^2\,dx = \left[x^3\right]_0^t = t^3$$

5. Invert the cumulative distribution function.

$$u = F(x) = x^3 \implies x = u^{1/3}$$

Therefore, an inverse CDF function can be written in MATLAB

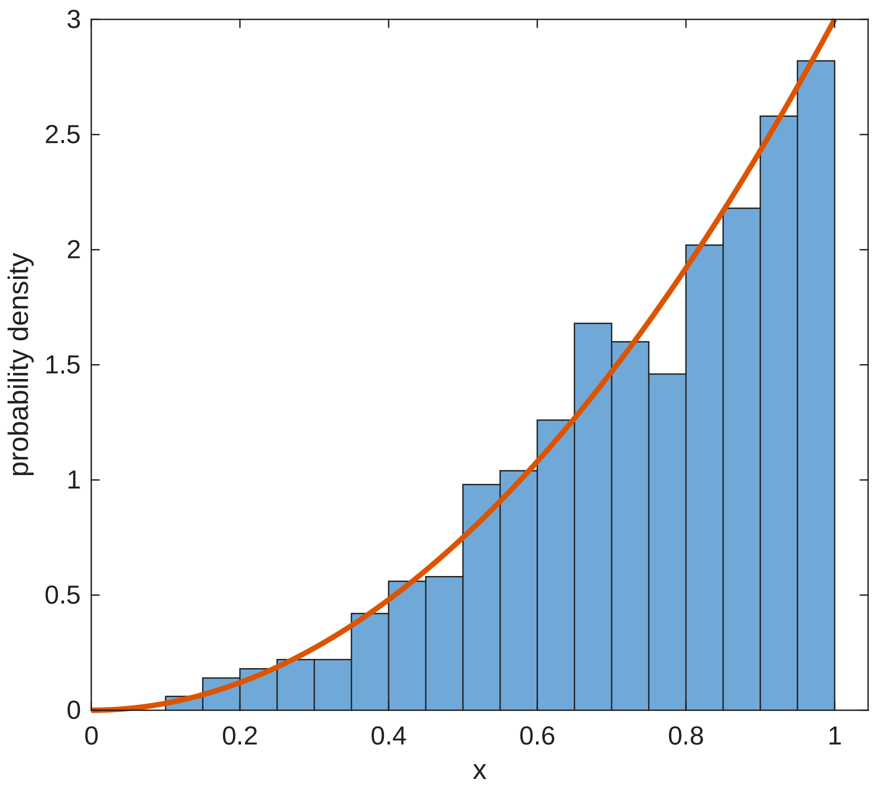
```
inv_cdf = @(u) u.^(1/3);
```

And it can be used to draw a single value from $f(x)$ as follows

```
% draw u from a uniform distribution on [0, 1]
u = rand;

% use the inverse CDF to obtain the corresponding x value
x = inv_cdf(u)
```

6. Your plot should look something like this:



The following script will sample 1000 x values and plot the figure

```
clearvars

% parameters
% ---

% number of samples to draw
```

```
nbr_samples = 1000;

% inverse CDF
inv_cdf = @(u) u.^(1/3);


% draw samples
% ---

% inverse transform sampling of X
u_samples = rand(1, nbr_samples);
x_samples = inv_cdf(u_samples);


% calculate PDF over X range
% ---

xs = linspace(0, 1);
fs = 3 * xs.^2;


% plot
% ---

histogram(x_samples, "Normalization", "pdf");
hold on
plot(xs, fs, "LineWidth", 2);
xlabel("x")
ylabel("probability density")
hold off
```

Question 19:

Define the random variable X as the profit from the scheme, i.e., resale value $-$ ticket cost. The script below calculates the quantities asked

```
clearvars

% parameters
% ---

ticket_cost = 80;
resale_values = [400; 80; 0];
p = [0.25; 0.35; 0.40];
```

```
% define random variable X: profit
x = resale_values - ticket_cost;

% check that the probability distribution if valid
% ---

assert(all(p >= 0), 'probabilities must be nonnegative');
assert(abs(sum(p) - 1) < 1e-10, 'probabilities must sum to 1');

% calculate expected value, variance
% ---

EX  = sum(x .* p);            % expected value
EX2 = sum(x.^2 .* p);
varX = EX2 - EX^2;            % variance
SD = sqrt(varX);              % standard deviation

% display
disp("Expected value: " + EX);
disp("Variance: " + varX);
disp("Standard deviation: " + SD);


% probability of no profit
% ---

% P(X <= 0)
p_noprofit = sum(p(x <= 0));
disp("Probability of making no profit: " + p_noprofit);
```

The outputs are:

```
>> prob_discrete
Expected value: 48
Variance: 25856
Standard deviation: 160.798
Probability of making no profit: 0.75
```

Space cows (Markov chain)

Question 20:

The sample space consists of the 3 possible locations of the cows:

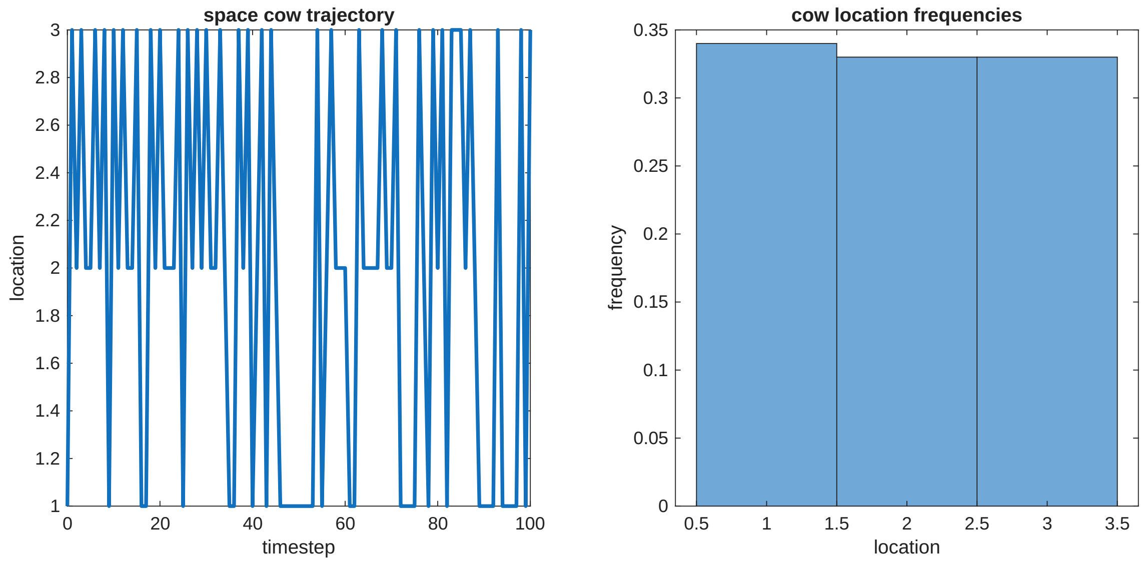
$$\{\text{field}, \text{waterhole}, \text{native habitat}\}$$

To be a proper transition matrix, the columns must sum to 1

$$T = \begin{bmatrix} 0.5 & 0.2 & 0.35 \\ 0.1 & 0.4 & 0.55 \\ 0.4 & 0.4 & 0.1 \end{bmatrix}$$

Question 21:

Your final figure should look something like this:



Below are two scripts that will generate the solution and figure.

First, a script that uses an `if` statement to randomly sample the next location:

```
clearvars

% parameters
% ---

% transition matrix (T_{i,j} is prob of transition j -> i)
T = [
   %  1    2     3 next locations
    0.5, 0.2, 0.35; % previous location 1
    0.1, 0.4, 0.55; % previous location 2
```

```
    0.4, 0.4, 0.1   % previous location 3
];

% number of steps to simulate the cow
nbr_steps = 100;

% initial location is the field
initial_location = 1;


% simulate location for nbr_steps
% ---

% each column is the CDF of the cow's destination given the previous location
cdfs = cumsum(T);

% preallocate a vector for the cow's locations
% index 1 is the initial location + nbr_steps subsequent locations
locations = zeros(1 + nbr_steps, 1);

% store cow's initial location
locations(1) = initial_location;

% simulate the cow for nbr_steps timesteps
for step_nbr = 1:nbr_steps

    % cow's previous location
    prev_location = locations(step_nbr);

    % CDF for the cow's next location given their previous location
    cdf = cdfs(:, prev_location);

    % perform inverse transform sampling to randomly select next location
    u = rand;
    if u < cdf(1)
        next_location = 1;
    elseif u < cdf(2)
        next_location = 2;
    else
        next_location = 3;
    end

    % store next location in storage array
    locations(step_nbr + 1) = next_location;
end
```

```
% plot
% ---

% plot trajectory panel
subplot(1, 2, 1);
plot(0:nbr_steps, locations, "LineWidth", 2);
xlabel("timestep")
ylabel("location")
title("space cow trajectory")

% plot histogram panel
subplot(1, 2, 2);
histogram(locations(2:end), "Normalization", "probability")
xlabel("location")
ylabel("frequency")
title("cow location frequencies")
```

Alternatively, you can write the inverse CDF as an anonymous function to use directly. Replace the `% simulate location for nbr_steps` section with the code below:

```
% simulate location for nbr_steps
% ---

% create an anonymous function that randomly samples the
% next location given a random number and the previous
% location
cdfs = cumsum(T);
inv_cdf = @(u, prev_location) find(cdfs(:, prev_location) >= u, 1); % <-- anon func

% preallocate a vector for the locations
% index 1 is the initial location + nbr_steps subsequent locations
locations = zeros(nbr_steps + 1, 1);

% store initial location
locations(1) = initial_location;

% step the cow nbr_steps times
for step_nbr = 1:nbr_steps

    % previous location
    prev_location = locations(step_nbr);

    % perform inverse transform sampling to randomly select destination
```

```
    u = rand;
    new_location = inv_cdf(u, prev_location);                        % <-- used here

    % store new location in storage array
    locations(step_nbr + 1) = new_location;
end
```

Question 22:

In the long term, the proportion of time that the cows will spend in each state will approach a stationary distribution. The stationary distribution is the eigenvector associated with the dominant eigenvalue, which is the eigenvalue equal to 1.

To find the eigenvalues and eigenvectors, we use the `eig` function

```
>> [V, D] = eig(T)
V =
   -0.6081   -0.7071   -0.1961
   -0.5891    0.7071   -0.5883
   -0.5321   -0.0000    0.7845

D =
    1.0000         0         0
         0    0.3000         0
         0         0   -0.3000
```

MATLAB has returned a solution with the dominant eigenvalue (equal to 1) in the first diagonal entry of `D`; therefore, the corresponding eigenvector is in the first column of `V`. A probability distribution must sum to 1, so to obtain the stationary distribution, we normalise the dominant eigenvector:

```
>> V(:, 1) / sum(V(:, 1))
ans =
    0.3516
    0.3407
    0.3077
```

Therefore, in the long term, the cows will spend approximately 35% of their time in the field, 34% of their time at the waterhole, and 31% of their time in the native habitat.