Week 2 - Maths refresher 2

Linear algebra & probability

the university
for the real world

Feedback

How are lectures/tutorials? Is there a topic you're particularly struggling with / want more support for?

Lecture 1 is now on Canvas

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Dashboard

Courses

Calendar

Inbox

History

Commons

Search

edia

2026 SEM-2

Home

Announcements

Syllabus

Modules

Assignments

Grades

Discussions

QUT Readings

Study support

Personal support

Library

People

Zoom

Collaborations

Search

Student Voice Surveys

Studiosity 24/7

1.1 Lecture

Maths refresher I: sets, functions and optimization

Welcome to the first week of MXB261!

This week, we review the basic mathematical language of game theory: sets, functions, and optimisation. This is important because games are formal models of choices and outcomes, and we need these tools before we can analyse or simulate strategic decision-making in MATLAB.

Where and when?

The lecture is held at **9-11am on Monday** in **GP-B-121**. If you are enrolled as an online student, you may attend via Zoom.

The meeting ID: 865 0851 8756

Passcode: 349848

Materials from the lecture:

- NOTES
  - Housekeeping: Week 1 Interactive Lecture Slides.pptx
  - Slides (final version): week_1_lecture_annotated.pdf
- VIDEO
  - Zoom recording Password: TEY*V$6q

Backup space

nadiah.org

Feedback: Slow down

Via Slido QR code:

> *Thank you, your slides are well prepared. But Could you please little bit slow down when you sharing slides.*

Thank you so much for giving me feedback.

I definitely went too fast, finished 20 minutes before I expected.

Feedback: What I will do

Things I'm going to try this lecture to slow myself down:

1. More writing on slides
    - Won't slow my speaking, but will slow the slides
2. Cues to repeat content of slide
    - If you miss it the first time, fingers crossed you'll catch it the the second time

MATLAB is slow / not working

MATLAB is running super slow:

1. Run in no-desktop mode. On Linux:

```
# add this line to your .bashrc
alias matlabcli='/usr/local/MATLAB/R2026a/bin/matlab -nodesktop
    -nojvm -nosplash'
```

2. On Windows: Visual Studio Code can run a MATLAB terminal.

Last year, 1 student couldn't get MATLAB running at all:

GNU Octave

Today's topics

1. Linear algebra
2. Probability
3. Markov chains
4. Solving ODEs (maybe, if time)

Linear algebra: Three perspectives

Central problem, the linear system:

$$A\mathbf{x} = \mathbf{b}$$

Example:

$$\overbrace{\begin{bmatrix} -1 & 1 \\ -2 & -1 \end{bmatrix}}^{A} \overbrace{\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}}^{\mathbf{x}} = \overbrace{\begin{bmatrix} 2 \\ 1 \end{bmatrix}}^{\mathbf{b}}$$

You'll be using MATLAB, so I want to give you a conceptual understanding. Three ways to think about:

1. Row picture
2. Column picture
3. Linear map

1. Row picture: system of equations

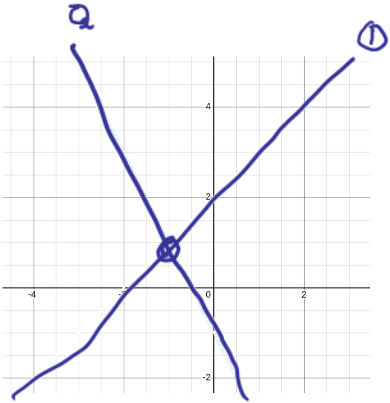
2×2 example in matrix form:

$$\begin{bmatrix} -1 & 1 \\ -2 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

?

As the intersection of two lines:

$$-x_1 + x_2 = 2 \quad ①$$
$$-2x_1 - x_2 = 1 \quad ②$$



1. Row picture: system of equations

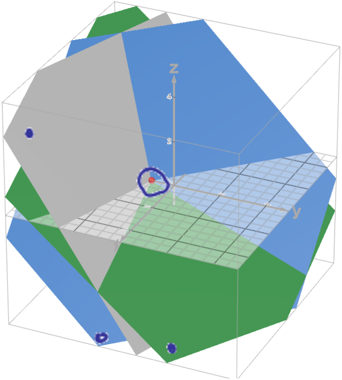
3×3 example in matrix form

$$\begin{bmatrix} 1 & 3 & 2 \\ 2 & 2 & 2 \\ 3 & 5 & 6 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -3 \\ -2 \\ -5 \end{bmatrix}$$

?

Intersection of three planes:

$$\begin{aligned} x + 3y + 2z &= -3 \\ 2x + 2y + 2z &= -2 \\ 3x + 5y + 6z &= -5 \end{aligned}$$

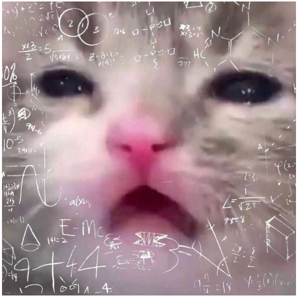


1. Row picture: system of equations

Higher-dimensional matrices

$$\begin{bmatrix} 2 & -2 & 2 & 9 \\ -3 & 3 & 2 & 5 \\ 3 & 5 & -6 & 6 \\ 1 & 8 & 6 & 8 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 5 \\ 2 \\ -5 \\ 8 \end{bmatrix}$$

Intersection of
higher-dimensional objects



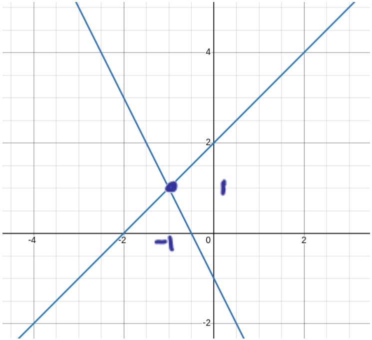
2. Column picture

Recall the row picture:

$$\begin{bmatrix} -1 & 1 \\ -2 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

Its solution is:

$$\underset{\sim}{x} = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$



2. Column picture

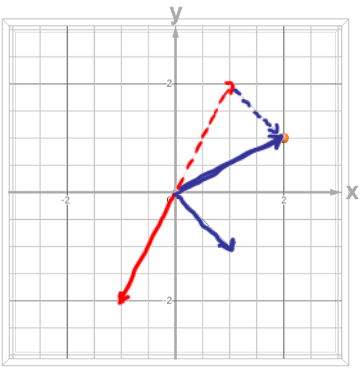
The equation

$$\begin{bmatrix} -1 & 1 \\ -2 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

can also be written:

$$\underset{-1}{x_1} \begin{bmatrix} -1 \\ -2 \end{bmatrix} + \underset{+1}{x_2} \begin{bmatrix} 1 \\ -1 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

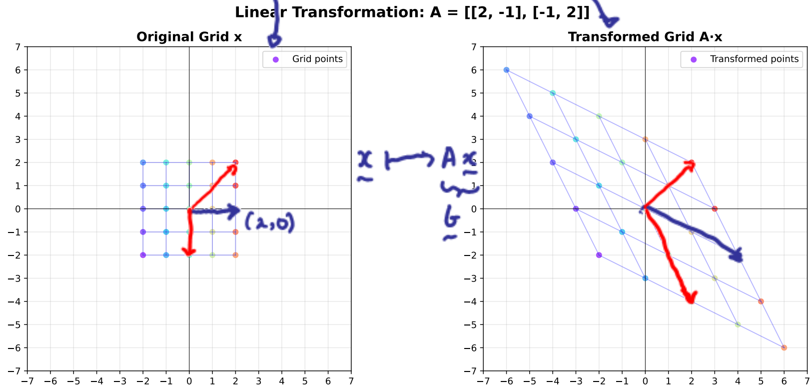
Tip-to-tail:



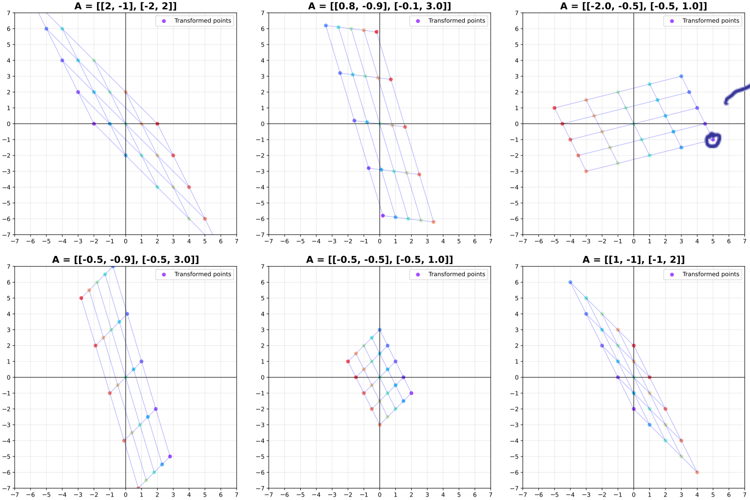
3. Linear map: $\mathbf{x} \mapsto A\mathbf{x}$

Example: $\overset{A}{\begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix}} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}$

$$\begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix} \begin{bmatrix} 2 \\ 0 \end{bmatrix} = \begin{bmatrix} 4 \\ -2 \end{bmatrix}$$



3. Linear map: examples



flip.

Matrix inverse

$$a x = b$$
$$x = \frac{b}{a}$$ inv.

If

$$Ax = b$$

?

then

$$x = A^{-1}b$$

(inverse

```
>> A = [2 -1; -1 2];
>> inv(A)
ans =
    0.6667    0.3333
    0.3333    0.6667
```

$= A^{-1}$

a

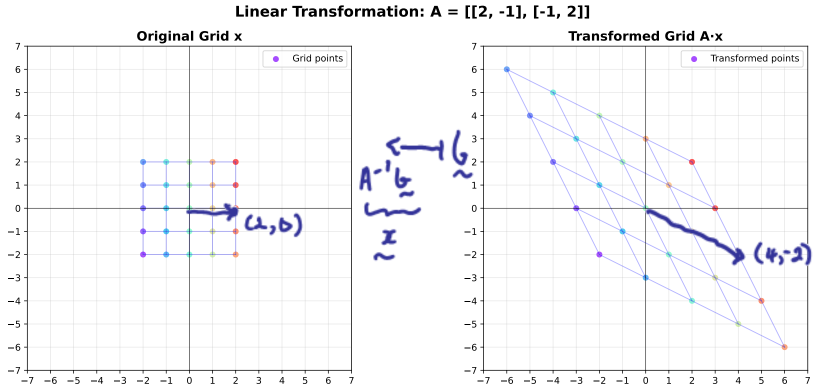
3. Linear map: inverse $\mathbf{b} \mapsto A^{-1}\mathbf{b}$

Example: $\underbrace{\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}}_{\underset{\sim}{x}} = \underbrace{\begin{bmatrix} 2/3 & 1/3 \\ 1/3 & 2/3 \end{bmatrix}}_{A^{-1}} \underbrace{\begin{bmatrix} b_1 \\ b_2 \end{bmatrix}}_{\underset{\sim}{b}}$

```
>> inv(A) * [4; -2]
ans =
     2
     0
```

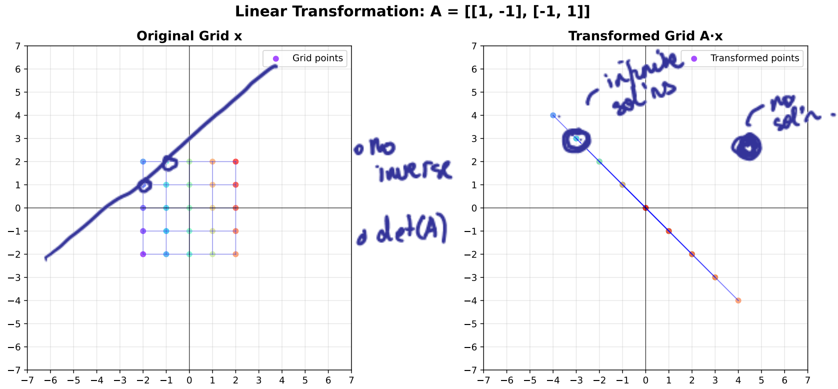
b

x

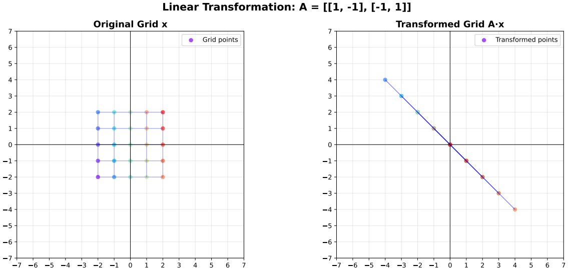


3. Linear map: ?

Discuss: What's happening with the inverse here?



3. Linear map: no inverse




```
>> A = [1 -1; -1 1];
>> inv(A)
Warning: Matrix is singular to working precision.
ans =
   Inf   Inf
   Inf   Inf
```

syn

Summary: Three $A\mathrm{x} = \mathrm{b}$ perspectives

1. Row picture:

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \ldots &= b_1 \\ a_{21}x_1 + a_{22}x_2 + \ldots &= b_2 \\ \vdots \end{aligned}$$

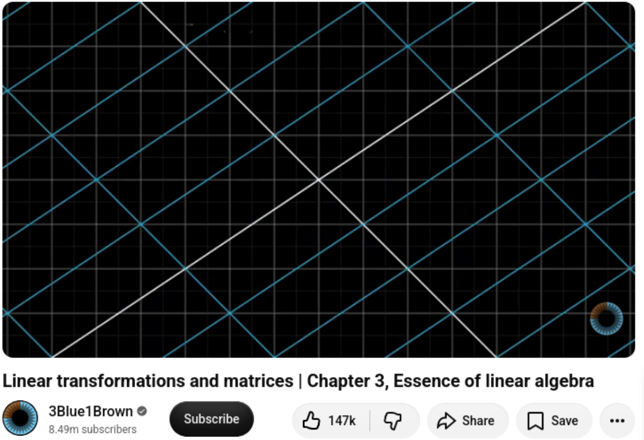
2. Column picture:

$$x_1 \begin{bmatrix} | \\ \text{col } 1 \\ | \end{bmatrix} + x_2 \begin{bmatrix} | \\ \text{col } 2 \\ | \end{bmatrix} + \ldots = \mathbf{b}$$

3. Linear map

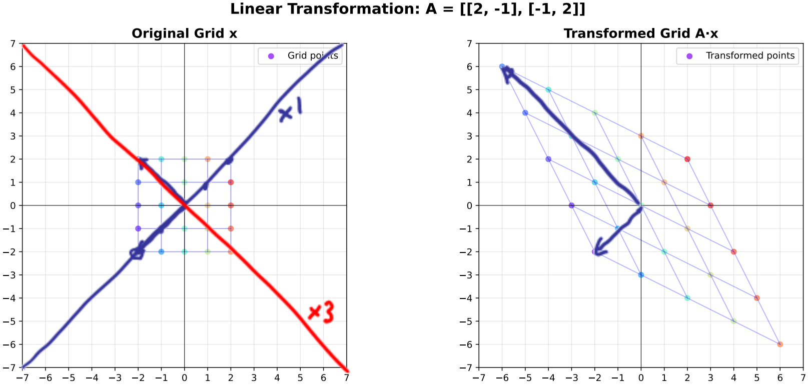
$$x \mapsto \underbrace{Ax}_{=b}$$

Highly recommend: 3Blue1Brown

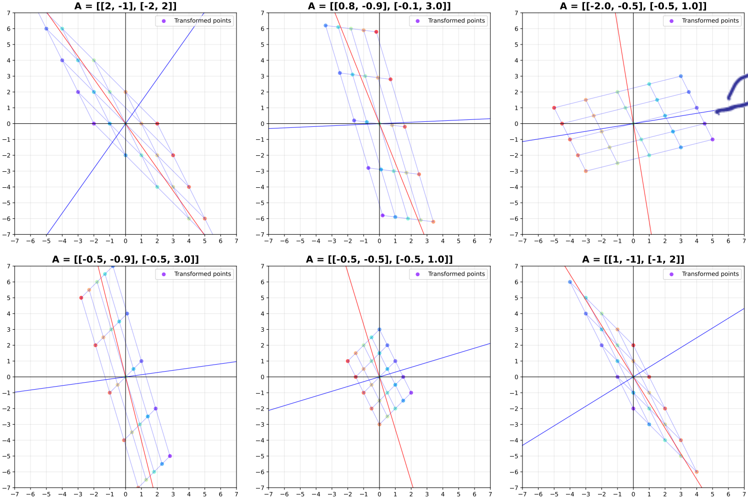


Special directions

Can you spot some vectors that don't change direction?

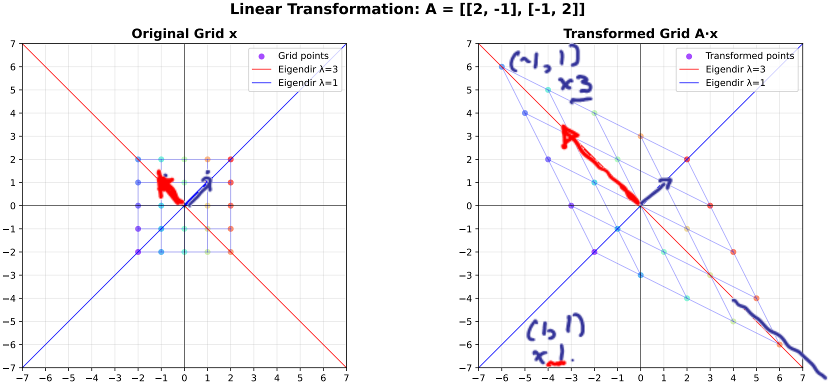


Special directions & values: an intrinsic property



mult by neg number

Intrinsic directions and values



Eigenvalue problem

eigen = 'intrinsic'

eigenvalue — special value

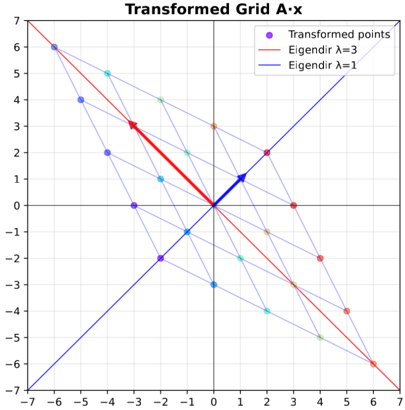
$$A\,\underbrace{\mathbf{v}}_{\text{eigenvector}} = \overbrace{\lambda}^{\text{eigenvalue}}\,\underbrace{\mathbf{v}}_{\text{eigenvector}}$$

eigenvector — special dir

Our example:

$$\begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix} \overset{\underset{\sim}{v}_1}{\begin{bmatrix} 1 \\ 1 \end{bmatrix}} = \overset{\lambda_1}{1} \overset{\underset{\sim}{v}_1}{\begin{bmatrix} 1 \\ 1 \end{bmatrix}}$$

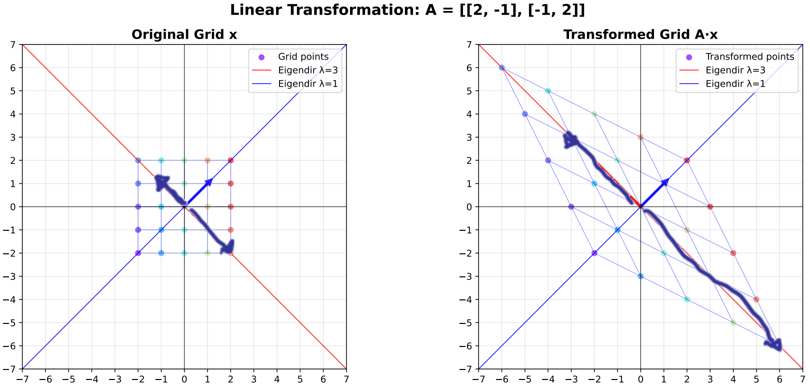
$$\begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix} \underset{\underset{\sim}{v}_2}{\begin{bmatrix} -1 \\ 1 \end{bmatrix}} = \underset{\lambda_2}{3} \underset{\underset{\sim}{v}_2}{\begin{bmatrix} -1 \\ 1 \end{bmatrix}}$$



Why I sometimes say 'eigendirection'

If $\mathbf{v}$ is an eigenvector, then so is $k\,\mathbf{v}$ for any scalar $k \neq 0$ (incl. $k < 0$).

With pen and paper, I found: $\lambda_2 = 3 \quad \underset{\sim}{v}_2 = (-1, 1)$



Eigenvalue problem with pen and paper

Find λ_i, $\mathbf{v}_i$ pairs that solve:

$$A \underset{1}{\mathbf{v}} = \overset{2}{\lambda} \underset{1}{\mathbf{v}}$$

Can we solve analytically for matrix size:

- 2×2?
- 3×3?
- 4×4?

possible

- higher?

impossible

Eigenvalue problem in MATLAB

$$A\,\mathbf{v} = \lambda\,\mathbf{v}$$

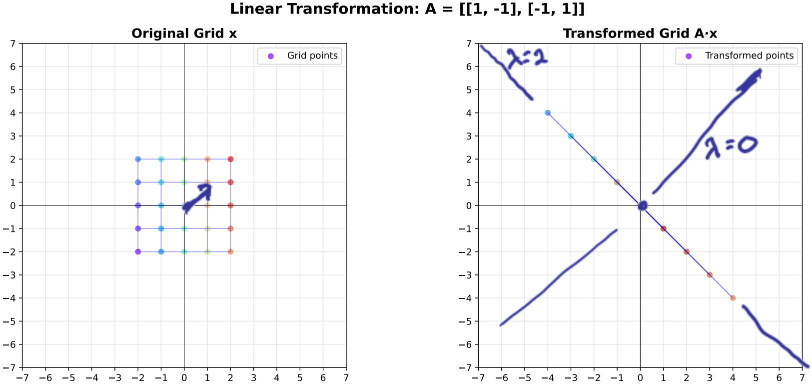
```
>> help eig
 eig - Eigenvalues and eigenvectors

    Syntax
      [V,D] = eig(A)

    Input Arguments
      A - Input matrix
        square matrix

    Output Arguments
      V - Right eigenvectors
        square matrix
      D - Eigenvalues (returned as matrix)
        diagonal matrix
```

$$V = \begin{bmatrix} | & | & | & \\ \underset{\sim}{v_1} & \underset{\sim}{v_2} & v_3 & \cdots \\ | & | & | & \end{bmatrix}$$

$$D = \begin{bmatrix} \lambda_1 & 0 & 0 & \cdots \\ 0 & \lambda_2 & 0 & \cdots \\ 0 & 0 & \lambda_3 & \cdots \\ \vdots & \vdots & \vdots & \end{bmatrix}$$

d

Question: Eigenvalues in this example?



Linear algebra: Summary

Inverse:

- If $A\mathbf{x} = \mathbf{b}$ then $\mathbf{x} = A^{-1}\mathbf{b}$
- In MATLAB: `x = inv(A) * b`
- Some matrices do not have an inverse
  - singular, zero eigenvalue, determinant equal to zero, linear system has no solution or infinite sol'ns.

Eigenvalue problem:

- Every matrix has $(\lambda, \mathbf{v})$ pairs that are intrinsic to that matrix, found by solving the eigenvalue problem $A\,\mathbf{v} = \lambda\,\mathbf{v}$
  - If $\mathbf{v}$ is an eigenvector, then so is $k\mathbf{v}$ for any scalar $k \neq 0$
- In MATLAB: `[V,D] = eig(A)`

vect values.

Probability: Principle of Indifference

The Principle:

> *If we have no grounds for distinguishing between possible outcomes, then we should favour them equally. And the extent of favouring can be represented by a probability.*

Example:

- Flip of two coins, outcomes: $\{\text{HH}, \text{HT}, \text{TH}, \text{TT}\}$
- Equal probability: $P(\text{HH}) = \frac{1}{4}$, $P(\text{HT}) = \frac{1}{4}$, and so on.

*Google: Principle of Indifference paradoxes

Notation

$$P(\overbrace{X}^{\text{random variable}} \leq \overbrace{x}^{\text{its value}})$$

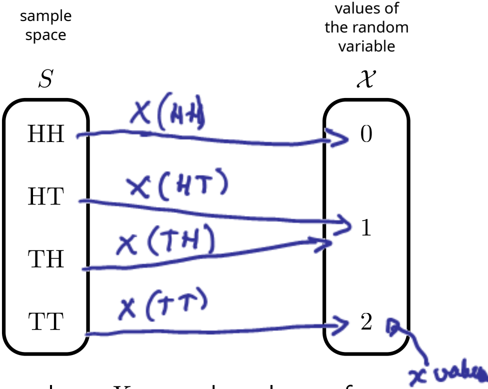
A random variable is a function

A **random variable** is a function that maps from outcomes to numbers (or elements of some set)

Example: Flipping two coins.

We could define a random variable X that counts the number of tails

- $X(HH) = 0$
- $X(HT) = 1$
- $X(TH) = 1$
- $X(TT) = 2$



X is the function, not the values themselves. X can take values x from the set of possible values $\mathcal{X}$.

Why is this useful?

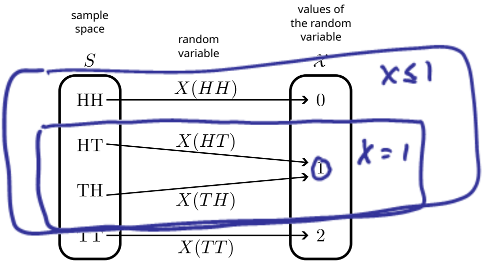
Once we have defined a random variable X, we can ask probability questions about it.

Example: Flip 2 coins, X is the number of tails

Probability questions:

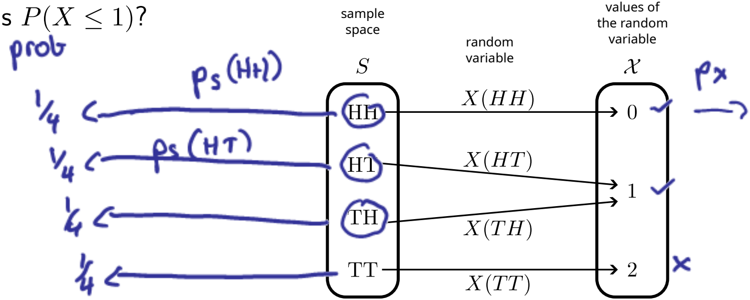
- $P(X = 1)$?
- $P(X \leq 1)$?
- $E[X]$?

↳ $E(x)$



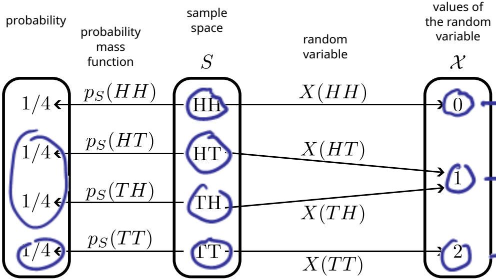
Example probability question about X

What is $P(X \leq 1)$?



$$
\begin{aligned}
P(X \leq 1) &= P(\{s \in S \mid X(s) \leq 1\}) \\
&= P(\{HH, HT, TH\}) \\
&= p_S(HH) + p_S(HT) + p_S(TH) \\
&= 1/4 + 1/4 + 1/4 = 3/4
\end{aligned}
$$

Usually, we deal with the PMF of X, not S



PMF Probability mass function.

$p_x(0) \rightarrow 1/4$

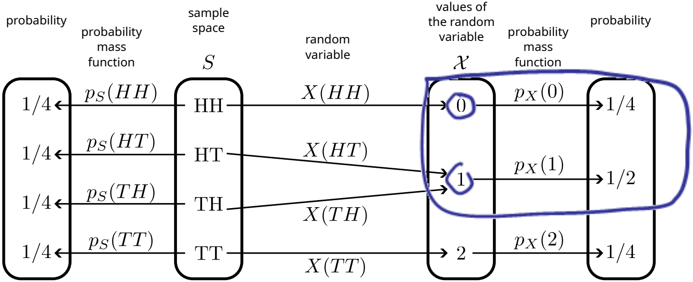
$p_x(1) \rightarrow 2/4 = 1/2$

$p_x(2) \rightarrow 1/4$

First, find PMF for X: $p_X(x) = P(\{s \in S \mid X(s) = x\})$

$$p_X(0) = P(\{HH\}) = p_S(HH) = 1/4$$
$$p_X(1) = P(\{HT, TH\}) = p_S(HT) + p_S(TH) = 1/2$$
$$p_X(2) = P(\{TT\}) = p_S(TT) = 1/4$$

Usually, we deal with the PMF of X, not S



Now when we ask, "What is $P(X \leq 1)$?", we can use p_X directly:

$$P(X \leq 1) = p_X(0) + p_X(1) = 1/4 + 1/2 = 3/4$$

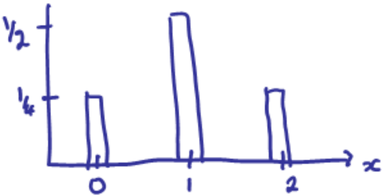
We often shorten $p_X(x)$ to just $p(x)$

Probability distributions

probability distributions

discrete

continuous



probability mass function (PMF)

probability density function (PDF)

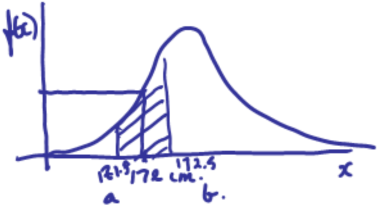
Two main requirements:

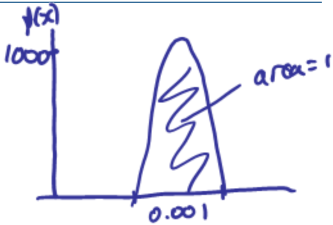
1. Non-negative: $p(x) \geq 0$
2. Sums to 1: $\sum_{x \in \mathcal{X}} p(x) = 1$

Two main requirements:

1. Non-negative: $f(x) \geq 0$
2. Integrates to 1: $\int_{-\infty}^{\infty} f(x)\, dx = 1$

Some notes on the PDF $f(x)$





- For a continuous distribution, $P(X = x) = 0$
  - What's the probability that a student's height is *exactly* 172.0000... cm?
- The PDF $f(x)$ does not give probabilities, it gives densities
  - e.g., it's possible to have $f(x) > 1$
  - That's why, to find probabilities, we must take the integral:

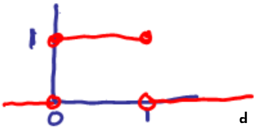
$$P(a \leq X \leq b) = \int_a^b f(x)\, dx$$

Notation

random variable — distributed as

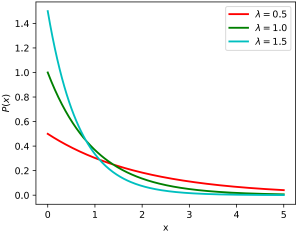
$$U \sim \mathrm{Unif}(0, 1)$$

name of distribution — parameters



Exponential distribution:

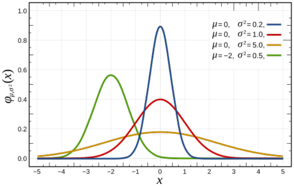
$$X \sim \mathrm{Exp}(\lambda)$$



CC, EvgSkv

Normal distribution:

$$X \sim \mathcal{N}(\mu, \sigma^2)$$



CC, Inductiveload

Discrete expected value and variance

x	1	2	3	4
p	0.1	0.2	0.3	0.4

For probability mass function:

$$P(X = x_i) = p_i$$

Expected value:

$$E[X] = \sum_i x_i\, p_i$$

Variance:

$$\mathrm{Var}(X) = \sum_i (x_i - E[X])^2 p_i = \sum_i x_i^2 p_i - \big(E[X]\big)^2$$

Standard deviation:

$$\sigma_X = \sqrt{\mathrm{Var}(X)}$$

Discrete example

- $P(X = x_i) = p_i$ probability mass function

```
x = [  1    2    3    4]; % x_i values
p = [0.1  0.2  0.3  0.4]; % p_i probabilities
```

- $E[X] = \sum_i x_i\, p_i$ dot : elementwise expected value

```
EX  = sum(x .* p);
```

- $\mathrm{Var}(X) = \sum_i x_i^2 p_i - \left(E[X]\right)^2$ variance

```
EX2 = sum(x.^2 .* p);
varX = EX2 - EX^2;
```

- $\sigma_X = \sqrt{\mathrm{Var}(X)}$ standard deviation

```
SD = sqrt(varX);
```

Continuous expected value and variance

For probability mass function:

$$P(a \leq X \leq b) = f(x)$$

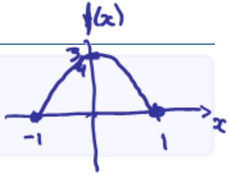
Expected value:

$$E[X] = \int_{-\infty}^{\infty} x f(x)\, dx$$

Variance:

$$\mathrm{Var}(X) = \int_{-\infty}^{\infty} x^2 f(x)\, dx - \big(E[X]\big)^2$$

Continuous example



$$f(x) = \frac{3}{4}\left(1 - x^2\right) \text{ on } [-1, 1]$$

Verify total probability:

$$\int_{-1}^{1} f(x)\,dx = \int_{-1}^{1} \frac{3}{4}\left(1 - x^2\right)\,dx$$

$$= \frac{3}{4}\left[x - \frac{x^3}{3}\right]_{-1}^{1}$$

$$= \frac{3}{4}\left[\underbrace{1 - \frac{1}{3}}_{2/3} - \underbrace{\left(-1 + \frac{1}{3}\right)}_{-2/3}\right]$$

$$= \frac{3}{\cancel{4}} \cdot \frac{4}{3} = 1.$$

Continuous example

$$f(x) = \frac{3}{4}\left(1 - x^2\right) \text{ on } [-1, 1]$$

Expected value:

$$E(X) = \int_{-1}^{1} x f(x)\, dx$$

$$= \frac{3}{4}\int_{-1}^{1} (x - x^3).\, dx$$

$$= \frac{3}{4}\left[\frac{x^2}{2} - \frac{x^4}{4}\right]_{-1}^{1}$$

$$= \frac{3}{4}\left[\frac{1}{2} - \frac{1}{4} - \left(\frac{1}{2} - \frac{1}{4}\right)\right] = 0$$

Continuous example

$$f(x) = \frac{3}{4}\left(1 - x^2\right) \text{ on } [-1, 1]$$

Variance:

$$\text{Var}(X) = \int_{-1}^{1} x^2 f(x)\, dx - \overbrace{E(X)}^{=0}{}^2$$

f

$$= \frac{3}{4}\int_{-1}^{1} x^2 - x^4 \cdot dx$$

$$= \frac{3}{4}\left[\frac{x^3}{3} - \frac{x^5}{5}\right]_{-1}^{1}$$

$$= \frac{3}{4}\left[\frac{1}{3} - \frac{1}{5} - \left(\frac{-1}{3} + \frac{1}{5}\right)\right]$$

$$= \frac{3}{4}\left[\frac{5}{15} - \frac{3}{15} - \left(-\frac{5}{15} + \frac{3}{15}\right)\right] = \frac{3}{4} \cdot \frac{4}{15} = \frac{1}{5}$$

Cumulative distribution function

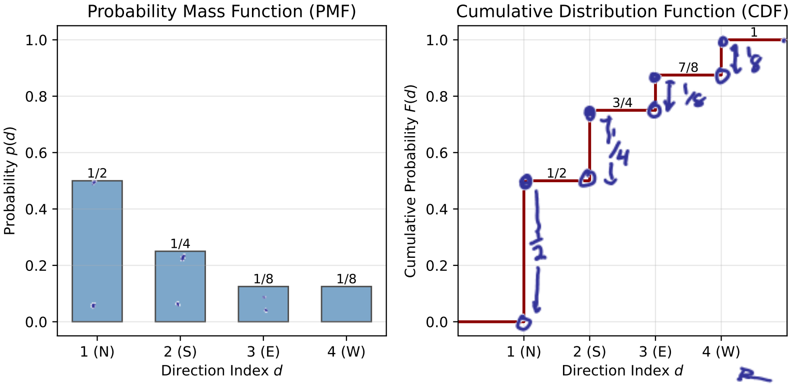
$$F(x) = P(X \leq x)$$

Discrete example

A random walk with different probabilities of stepping north, south, east, and west.

N → 1
S → 2
E → 3
W → 4.

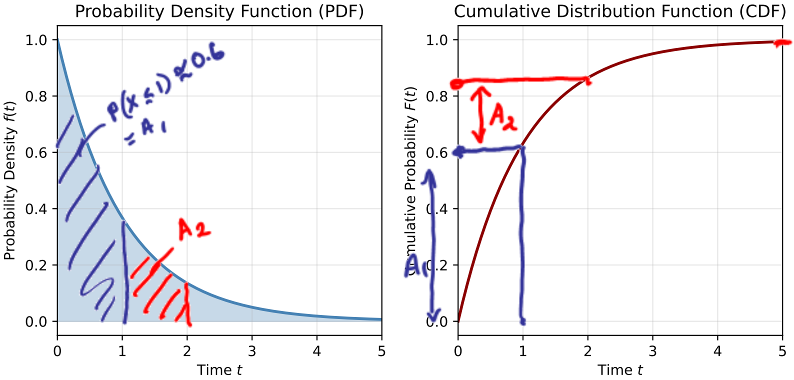
Define random variable, direction index D



$P(X \leq x)$

Continuous example

Waiting time between events



Cumulative distribution: discrete

Discrete:

$$F(x) = P(X \leq x)$$

Discrete example:

```
>> pmf = [1/2, 1/4, 1/8, 1/8]; % probability mass function
>> cdf = cumsum(pmf)
cdf =
    0.5000    0.7500    0.8750    1.0000
```

$\frac{1}{2} \to +\frac{1}{2} \to \frac{1}{8}$

Cumulative distribution: continuous

Continuous:

$$F(x) = P(X \leq x) = \int_{-\infty}^{x} f(t).dt$$

Continuous example:

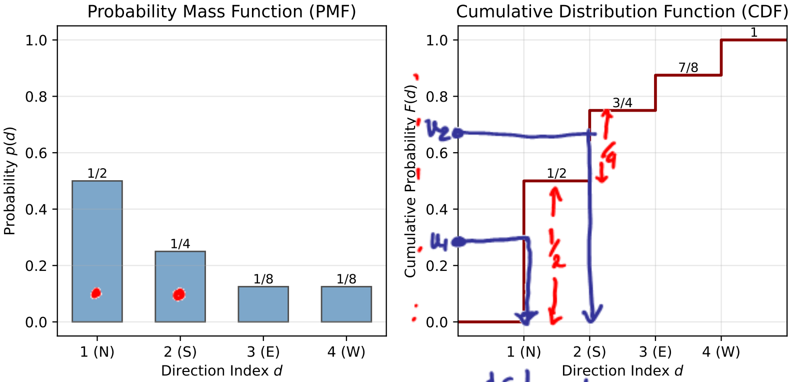
$$f(t) = \lambda e^{-\lambda t} \text{ for } t \geq 0$$

$$F(x) = \int_0^x \lambda e^{-\lambda t} = \left[\frac{\cancel{\lambda} e^{-\lambda t}}{-\cancel{\lambda}}\right]_0^x = -e^{-\lambda x} - (\underbrace{-e^0}_{-1})$$

$$= 1 - e^{-\lambda x}$$

Inverse transform sampling method

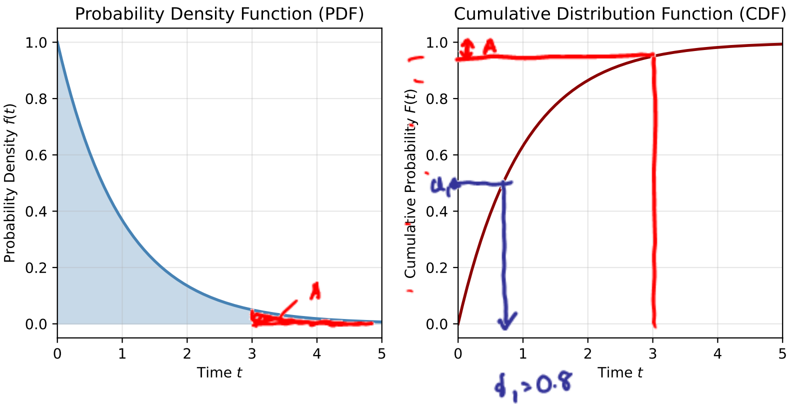
If F is the target CDF, then $X = F^{-1}(U)$ with $U \sim \mathrm{Unif}(0, 1)$, gives samples from the distribution with CDF F.
↳ u = rand;



d=1 d=2.

Inverse transform sampling method

Continuous example: exponential distribution, waiting time between events, $f(t) = \lambda e^{-\lambda t}$ and $F(t) = 1 - e^{-\lambda t}$ for $t \geq 0$.

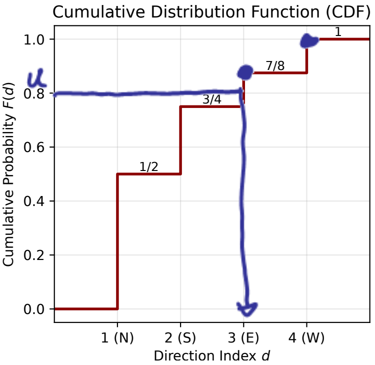


Discrete example

```
pmf   = [1/2, 1/4, 1/8, 1/8]; % probability mass function
dirns = ["N", "S", "E", "W"]; % directions
          1    2    3    4

cdf = cumsum(pmf); % create inverse transform sampling fnc
inv_cdf = @(u) find(cdf >= u, 1);      anonymous fnc.
```

```
>> cdf
cdf =
   1/2    3/4    7/8    1
>> u = 0.8; % u = rand;
>> cdf >= u
ans =
  1x4 logical array
   0    0    1    1
>> find(cdf >= u)
ans =
     3      4
>> find(cdf >= u, 1)
ans =
     3
```

1st value



d = 3.

Discrete example continued

```
pmf   = [1/2, 1/4, 1/8, 1/8]; % probability mass function
dirns = ["N", "S", "E", "W"]; % directions
                                step = [[+0, +1], [+0, -1] ... ]
cdf = cumsum(pmf); % create inverse transform sampling fnc
inv_cdf = @(u) find(cdf >= u, 1);

for step_nbr = 1:5 % display 5 steps
    u = rand;                      % draw uniform random value
    dirn_idx = inv_cdf(u);         % convert to direction index
    disp(dirns(dirn_idx));         % display direction
end
```

Example run:

```
>> sample_dirns                D
N                          1
W                          4
N                          1
E                          3
N                          1
```

Continuous example

$$F(x) = 1 - e^{-\lambda x}$$

Invert the function:

$$u = 1 - e^{-\lambda x}$$

$$x = \frac{-\ln(1-u)}{\lambda}$$

```
>> lambda = 1;
>> inv_cdf = @(u) -log(1 - u) / lambda;
>> u_samples = rand(5, 1);
>> x_samples = inv_cdf(u_samples)
x_samples =
    0.2613
    0.8016
    0.8923
    1.6764
    1.6924
```

5 std uniform.

Inverse transform sampling method: steps

Let $U \sim \text{Unif}(0, 1)$. For a random variable X with CDF F, the random variable $F^{-1}(U)$ has the same distribution as X.

To sample a value x:

1. Define F^{-1}

```
inv_cdf = @(u) ...;
```

2. Draw u from the standard uniform distribution

```
u = rand;
```

3. Convert u into x

```
x = inv_cdf(u);
```

Probability summary

- Principle of Indifference
- Some new notation, e.g., $P(X \leq x)$, $X \sim \mathrm{Exp}(\lambda)$
- A random variable is a function that maps outcomes $\rightarrow$ numbers
- Discrete and continuous probability distributions, and 2 requirements for both to be valid
- Calculating the expected value and variance
- Deriving the cumulative distribution function
- Inverse transform sampling method

Markov chains

Markov property: Future state only depends on previous state, not the sequence of states that preceded it (e.g., random walk)

For a discrete-time Markov process

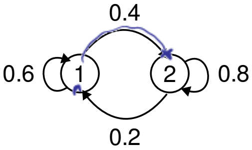
$$P(X_{t+1} = x \mid X_t = x_t, \ldots, X_1 = x_1) = P(X_{t+1} = x \mid X_t = x_t)$$

state in prev step.

Markov chain: discrete time and discrete state

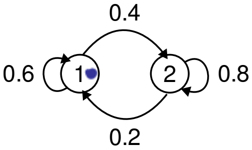
Example: Tree-stump game

Consider a child jumping between two tree stumps:



Does this have the Markov property?

Tree-stump game: Transition matrix



We could store the probabilities in a matrix

$$A = \begin{array}{c} \begin{array}{cc} \text{from 1} & \text{from 2} \end{array} \\ \begin{bmatrix} 0.6 & 0.2 \\ 0.4 & 0.8 \end{bmatrix} \end{array} \begin{array}{l} \\ \text{to 1} \\ \text{to 2} \end{array}$$

Tree-stump game: Simulation

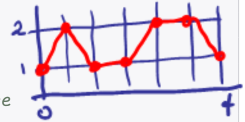
$$
\begin{array}{cc}
\text{loc1} & \text{loc2} \\
\downarrow & \downarrow
\end{array}
$$

$$
\text{cdfs} = \begin{bmatrix} 0.6 & 0.4 \\ 1 & 1 \end{bmatrix}
$$

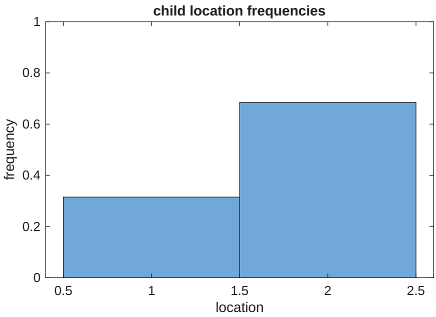
```
T = [0.6, 0.2; 0.4, 0.8];

% inv_cdf
cdfs = cumsum(T);
inv_cdf = @(u, prev_location) find(cdfs(:, prev_location) >= u, 1);

locations = [1; zeros(1000, 1)]; % preallocate
for step_nbr = 1:1000
    prev_location = locations(step_nbr);
    u = rand; % inverse transform sampling
    new_location = inv_cdf(u, prev_location);
    locations(step_nbr + 1) = new_location; % store
end

% plot histogram
histogram(locations(2:end), "Normalization", "probability")
ylim([0, 1])
xlabel("location"); ylabel("frequency"); title("child location
     frequencies")
```

← storage.

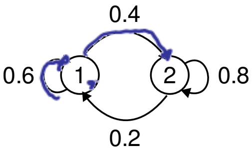
plot (locations)



Tree-stump game: Simulation



Tree-stump game: Future distribution



Example - child starts on stump 1. We predict 1 jump into the future:

$$\begin{array}{r} \text{to 1}\\ \text{to 2}\end{array}\overset{\text{from 1}}{\begin{bmatrix}0.6 & 0.2\\ 0.4 & 0.8\end{bmatrix}}\overset{\text{loc 1.}}{\underset{x_0}{\begin{bmatrix}1\\ 0\end{bmatrix}}} = \overset{x_1}{\begin{bmatrix}0.6\\ 0.4\end{bmatrix}}\begin{array}{l}\text{loc 1}\\ \text{loc 2.}\end{array}$$

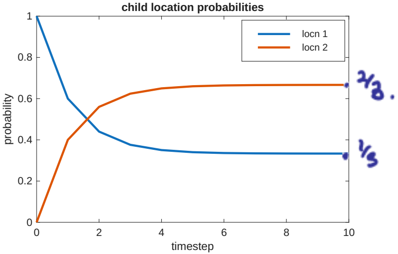
Two jumps into the future:

$$\underset{T}{\begin{bmatrix}0.6 & 0.2\\ 0.4 & 0.8\end{bmatrix}}\underset{T}{\begin{bmatrix}0.6 & 0.2\\ 0.4 & 0.8\end{bmatrix}}\underset{x_0}{\begin{bmatrix}1\\ 0\end{bmatrix}} = \overset{T}{\begin{bmatrix}0.6 & 0.2\\ 0.4 & 0.8\end{bmatrix}}\overset{x_1}{\begin{bmatrix}0.6\\ 0.4\end{bmatrix}} = \begin{bmatrix}0.44\\ 0.56\end{bmatrix}\begin{array}{l}\text{loc 1}\\ \text{loc 2.}\end{array}$$

$$x_0 \longrightarrow T^2 x_0 = x_2$$

Tree-stump game: Future distribution

```
>> x0 = [1; 0];
>> T^2 * x0
ans =
    0.4400
    0.5600

>> T^3 * x0
ans =
    0.3760
    0.6240
```



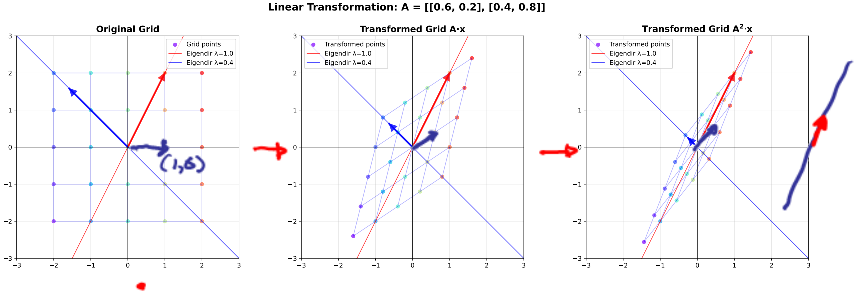
∞ jumps into the future?

$$\begin{bmatrix}0.6 & 0.2\\0.4 & 0.8\end{bmatrix}\begin{bmatrix}0.6 & 0.2\\0.4 & 0.8\end{bmatrix}\cdots\begin{bmatrix}0.6 & 0.2\\0.4 & 0.8\end{bmatrix}\underbrace{\begin{bmatrix}1\\0\end{bmatrix}}_{x_0} \to \boxed{\overset{T}{\begin{bmatrix}0.6 & 0.2\\0.4 & 0.8\end{bmatrix}}\overset{\underline{v}}{\begin{bmatrix}1/3\\2/3\end{bmatrix}} = \overset{\underline{v}}{\begin{bmatrix}1/3\\2/3\end{bmatrix}} \quad (\lambda = 1}$$

Example: Tree-stump game

$$T = \begin{bmatrix} 0.6 & 0.2 \\ 0.4 & 0.8 \end{bmatrix}$$ has two eigenvalue eigenvector pairs:

1. $\lambda_1 = 0.4$, $\mathbf{v}_1 = (-1, 1)$
2. $\lambda_2 = 1$, $\mathbf{v}_2 = (1, 2)$ DOMINANT



Find the stationary distribution

Solve the eigenvalue problem

$$T\mathbf{x}^* = \mathbf{x}^*$$

```
>> T = [0.6 0.2; 0.4 0.8];
>> [V, D] = eig(T)
V =
   -0.7071   -0.4472
    0.7071   -0.8944

D =
    0.4000         0
         0    1.0000

>> V(:, 2) / sum(V(:, 2))
ans =
    0.3333
    0.6667
```

normalise

1/3 2/3.

Markov chain: Summary

The stationary distribution of a Markov chain solves an eigenvalue problem

$$T\mathbf{x}^* = \mathbf{x}^*$$

Provided that the chain is:

- Irreducible: every state can reach every other state eventually
- Aperiodic: chain doesn't get stuck in fixed-period cycles

Preview: Ordinary differential equations

Examples I will use: population growth models and initial value problem

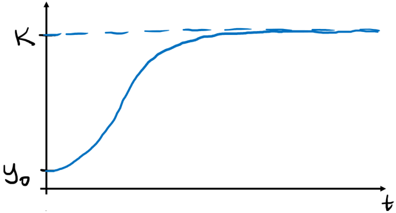
Let $y(t)$ be the number of individuals in a population at time t

Unconstrained growth model:

$$\frac{dy}{dt} = \lambda y$$

Constrained growth model:

$$\frac{dy}{dt} = \lambda y \left(1 - \frac{y}{K}\right)$$

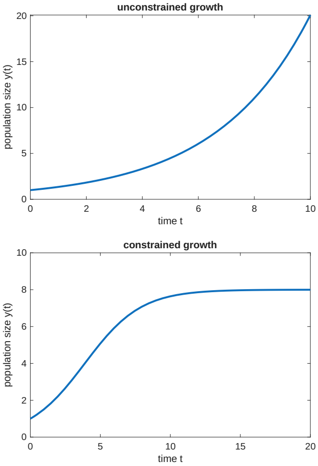


Initial value problem in MATLAB

Given that the intial population size is $y(0) = 1$, solve $y(t)$.

For unconstrained growth:

```
% parameters
y0 = 1; % initial pop size
lambda = 0.3;
t_span = [0 10]; % soln timespan

% differential equation
dydt = @(t, y) lambda * y;

% solve using ode45
[t, y] = ode45(dydt, t_span, y0);

% plot
plot(t, y, "LineWidth", 2)
```

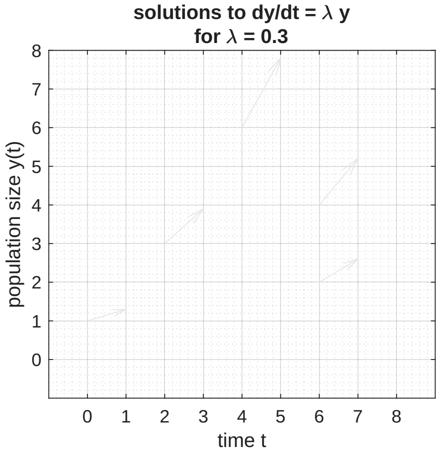


Quiver plot

Visual tool for understanding system dynamics.

Dynamics: $\dfrac{dy}{dt} = \lambda y$

Parameter: $\lambda = 0.3$



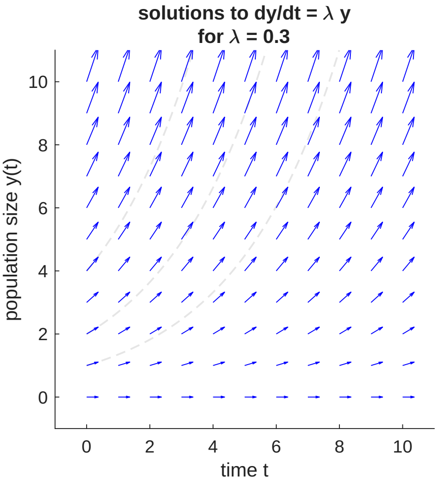
Example: unconstrained growth

Dynamics: $\dfrac{dy}{dt} = \lambda y$

Parameter: $\lambda = 0.3$

Initial values:

- $y(0) = 1$
- $y(0) = 2$
- $y(0) = 4$
- $y(0) = 0$



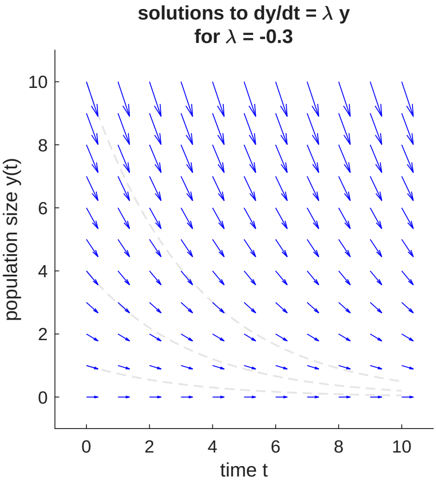
Example: population decay ($\lambda < 0$)

Dynamics: $\dfrac{dy}{dt} = \lambda y$

Parameter: $\lambda = -0.3$

Initial values:

- $y(0) = 10$
- $y(0) = 4$
- $y(0) = 2$
- $y(0) = 0$



Example: constrained growth

Dynamics:

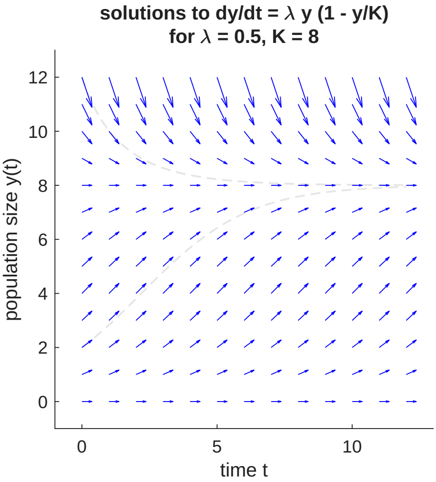
$$\frac{dy}{dt} = \lambda y \left(1 - \frac{y}{K}\right)$$

Parameters:

$$\lambda = 0.5 \text{ and } K = 8$$

Initial values:

- $y(0) = 0$
- $y(0) = 2$
- $y(0) = 8$
- $y(0) = 12$

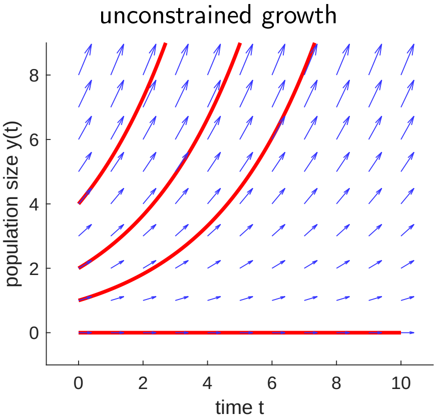


Stability

- Ball on a landscape analogy
  - Ball position = system state
  - Gravity = system dynamics
- **Steady state** (or equilibrium): ball at rest
  - Balanced on hilltop or resting at bottom of valley, both are at steady state
- **Stable** steady state: ball at bottom of valley
  - Small perturbations $\rightarrow$ system returns to steady state
- **Unstable** steady state: ball on hilltop
  - Small perturbations $\rightarrow$ system moves away from steady state

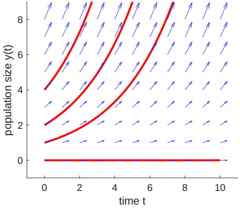
Example: steady states and stability

- Steady state
- Stable or unstable?



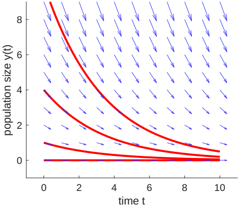
List steady states and classify

(a) unconstrained growth



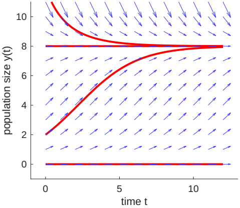
1. Steady state $y^* = 0$
   Unstable

(b) population decay



1.

(c) constrained growth



1.

2.

Mathematical definition: steady state

- Steady state: no change in time
- Find y^* that solves

$$\left.\frac{dy}{dt}\right|_{y=y^*} = 0$$

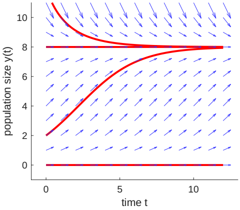
Example (c): $\frac{dy}{dt} = 0.5\, y \left(1 - \frac{y}{8}\right)$

- Find y^* that solves

$$\left.\frac{dy}{dt}\right|_{y=y^*} = 0.5\, y^* \left(1 - \frac{y^*}{8}\right) = 0$$

- Steady states: $y^* = 0$ or $y^* = 8$

(c) constrained growth



questions?

Stability test

A steady state y^* is stable if

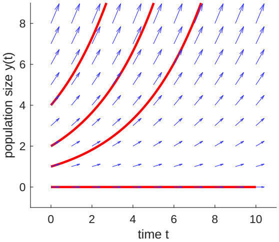
$$\frac{d}{dy}\left(\frac{dy}{dt}\right)\bigg|_{y=y^*} < 0$$

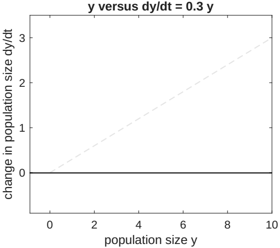
Example: stability in unconstrained growth

1. Plot dy/dt versus y

$$\frac{dy}{dt} = 0.3\ y$$

2. Draw dynamics on x-axis
   2.1 Arrows: direction of change in y
      - Positive change:
      - Negative change:
   2.2 Identify steady states and types
      - Unstable:
      - Stable:



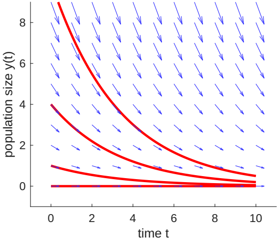


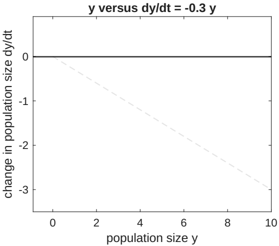
Example: stability in population decay

1. Plot dy/dt versus y

$$\frac{dy}{dt} = -0.3\,y$$

2. Draw dynamics on x-axis
   2.1 Arrows: direction of change in y
   2.2 Identify steady states and types



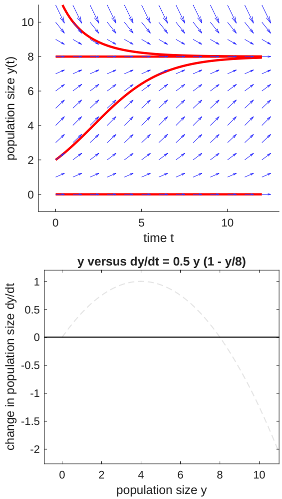


Example: stability in constrained growth

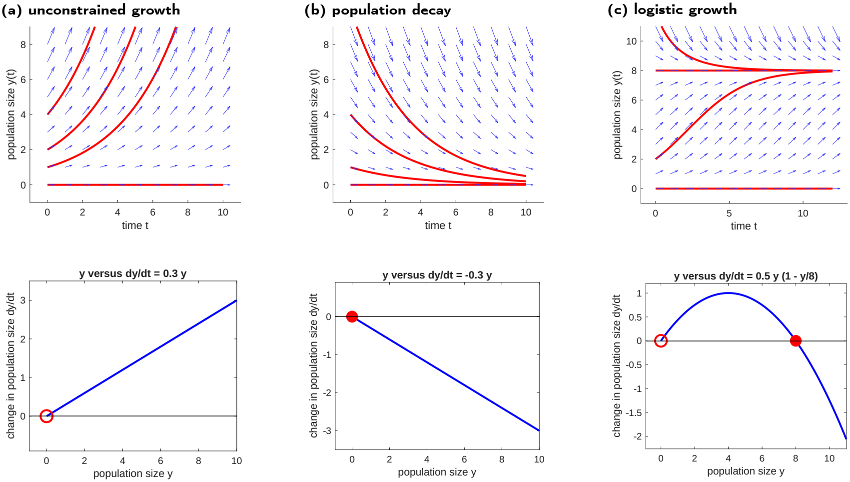
1. Plot dy/dt versus y

$$\frac{dy}{dt} = 0.5\, y \left(1 - \frac{y}{8}\right)$$

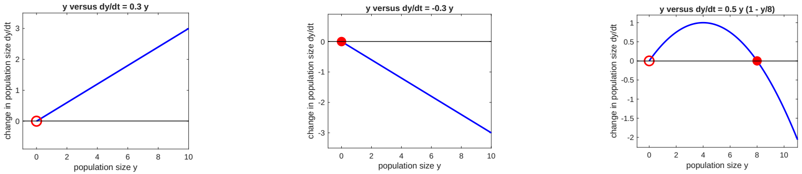
2. Draw dynamics on x-axis
   2.1 Arrows: direction of change in y
   2.2 Identify steady states and types



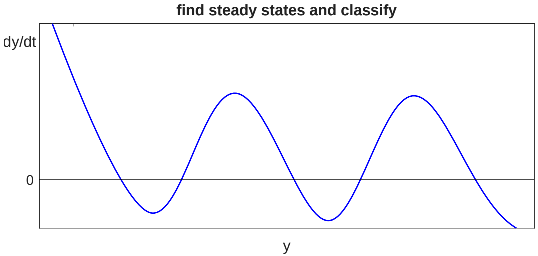
Examples together



Find and classify steady states

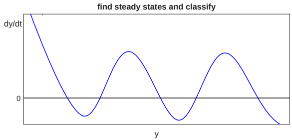


Find and classify steady states. What distinguishes stable from unstable? (Hint: look at slope of blue curve at steady states)



Steady states classification

- Positive slope: unstable
- Negative slope: stable
- Which slope, in maths?

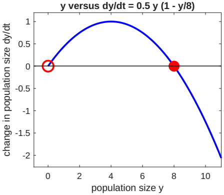


- **Stable** steady state:
  - Slope of dy/dt with respect to y at y^* is negative
  - $\frac{d}{dy}\left(\frac{dy}{dt}\right)\Big|_{y=y^*} < 0$
- **Unstable** steady state:
  - Slope of dy/dt with respect to y at y^* is positive
  - $\frac{d}{dy}\left(\frac{dy}{dt}\right)\Big|_{y=y^*} > 0$

qs?

Example: stability in constrained growth

- Dynamics: $\dfrac{dy}{dt} = 0.5\, y \left(1 - \dfrac{y}{8}\right)$
- Steady states: $y^* = 0$ and $y^* = 8$
- Stable if: $\dfrac{d}{dy}\left(\dfrac{dy}{dt}\right)\Bigg|_{y=y^*} < 0$



Rearrange for convenience:

$$\frac{dy}{dt} = \frac{y}{2} - \frac{y^2}{16}$$

Take derivative wrt y

$$\frac{d}{dy}\left(\frac{dy}{dt}\right) = \frac{1}{2} - \frac{y}{8}$$

When $y^* = 0$:

$$\frac{d}{dy}\left(\frac{dy}{dt}\right)\Bigg|_{y=0} = \frac{1}{2} > 0 \text{ Unstable}$$

When $y^* = 8$:

$$\frac{d}{dy}\left(\frac{dy}{dt}\right)\Bigg|_{y=8} = -\frac{1}{2} < 0 \text{ Stable}$$

Questions?

Feedback QR code for this week:

Break

Suggested questions:

1. What's your favourite band?
2. What's your group-project energy: plan everything 3 weeks early or thrive gloriously in the final 48 hours?

(There will be an exam on your friends' answers in Week 7)