The evolution of human cooperation: homophily, non-additive benefits, and higher-order relatedness

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Acknowledgements

Country:

I acknowledge the Turrbal and Yugara people and as the owners of this land. I pay respect to their Elders, past and present, and recognise this land has always been a place of teaching, learning, and research.

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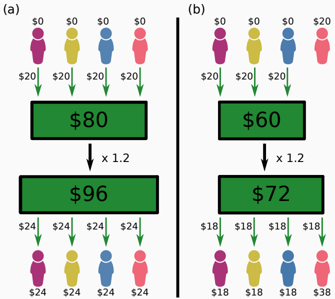
Public goods game

- Example:
  1. Four individuals
  2. Project multiples contributions by 1.2
  3. Returns are split equally

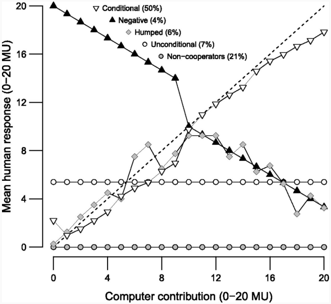


- Group-game version of Prisoner's Dilemma
  - Max total payoffs if everyone contributes
  - But max individual payoff if you don't contribute
    - Marginal per-capita return $= 1.2/4 = 0.3 < 1$
    - i.e., 30c back per \$1 contributed
- Never makes sense to contribute

How do people really behave in linear PGGs?

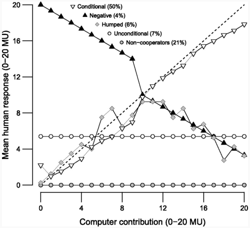
- Example: Burton-Chellew *et al.* (2016, PNAS)
  - Elicited contributious in PGG
  - Played against a computer
  - Computer play presumably removed fairness/empathy considerations

- Contribution level depends on contribution of others
- Similar results in other studies
- People genuinely seem believe this is payoff maximising!



Why do people make this mistake?

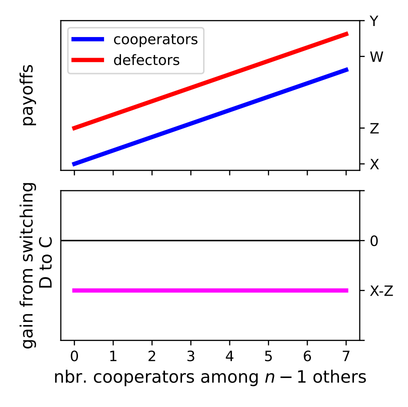
- Deeply unnatural scenario
- Previous work has focused on two 'mistakes':
    1. Mistake one-shot game for iterated game
    2. Mistake anonymous game for one with reputation concerns



My focus: Mistaking a linear game for a nonlinear one

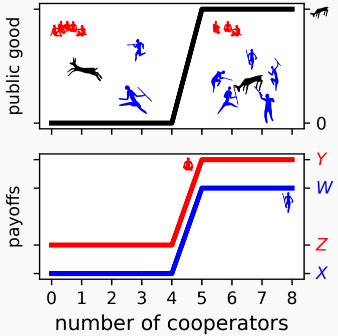
Linear public goods game

- In a linear game:
  - Benefit increases at constant rate with nbr. cooperators
  - No matter how many cooperators in the group, always lose by switching C to D
- n-player generalisation of PD



Nonlinear public goods game

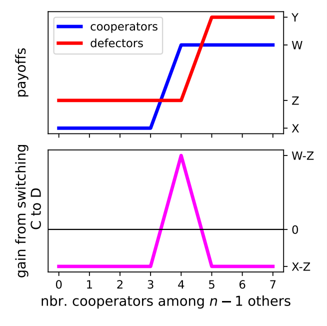
- Claim: sigmoid-shaped benefit functions particularly relevant to our early history
- Defectors always get higher payoffs than Cooperators



Nonlinear public goods game

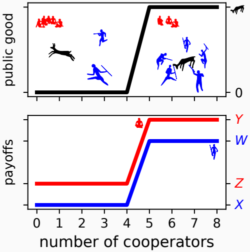
- Claim: sigmoid-shaped benefit functions particularly relevant to our early history

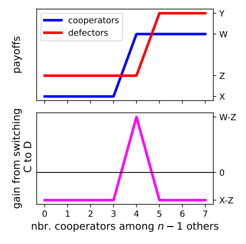
- Defectors always get higher payoffs than Cooperators
- However, if you are in a group that's one cooperator short of the threshold, you should cooperate
- In general:
    - if cooperators rare, don't cooperate
    - if cooperators common, might get higher payoffs if you're a cooperator



Nonlinear public goods game: evolutionary perspective

- In general:
  - if cooperators rare, don't cooperate
  - if cooperators common, might get higher payoffs if you also cooperate
- Evolutionary perspective:
  - if cooperators rare (invasion), cooperation can't succeed
  - if cooperators common, cooperation might persist





Embed the game in evolutionary dynamics

Replicator dynamics approach:

- Strategies (cooperate, defect) genetically encoded
- Clonal reproduction in an infinite population
- Higher payoff in the game $\rightarrow$ higher reproductive success

Replicator dynamics

Change in proportion of x-strategists:

m is nbr. strategies

$$\dot{p}_x = \underbrace{p_x}_{\text{proportion of } x\text{-strategists}} \left(\overbrace{\bar{\pi}_x}^{\text{expected payoff to } x\text{-strategists}} - \underbrace{\sum_{i=1}^{m} p_i \bar{\pi}_i}_{\text{expected payoff in population}} \right)$$

- growth rate proportional to how much better x-strategists' payoffs are compared to average

Replicator dynamics well-mixed

$$\dot{p}_x = p_x \left(\overbrace{\bar{\pi}_x}^{\text{expected payoff to } x\text{-strategists}} - \sum_{i=1}^{m} p_i \bar{\pi}_i \right)$$

- $\boldsymbol{e}_x$: indicator, focal plays strategy x (below: 1 when cooperator)
- g_{nf}: non-focal strategy distribution (below: nbr. cooperators among nonfocals)

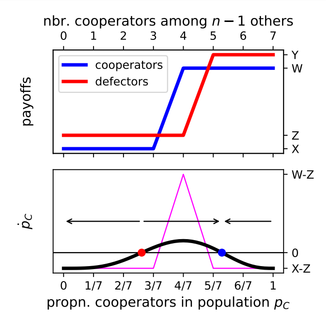
For prehistoric-hunt game:

$$\begin{aligned}
\bar{\pi}_C &= \sum_{g_{\mathrm{nf}}=0}^{n-1} \overbrace{\pi(e_C, g_{\mathrm{nf}})}^{\text{payoff}} \; \overbrace{\mathbb{P}[G_{\mathrm{nf}} = g_{\mathrm{nf}}]}^{\text{probability } g_{\mathrm{nf}} \text{ non-focals are cooperators}}, \\
&= \sum_{g_{\mathrm{nf}}=0}^{n-1} \pi(e_C, g_{\mathrm{nf}}) \; \overbrace{\binom{n-1}{g_{\mathrm{nf}}} p_C^{g_{\mathrm{nf}}} (1 - p_C)^{n-1-g_{\mathrm{nf}}}}^{\text{binomial}}
\end{aligned}$$

Recommend: Peña et al. (2014, J Theor Biol)

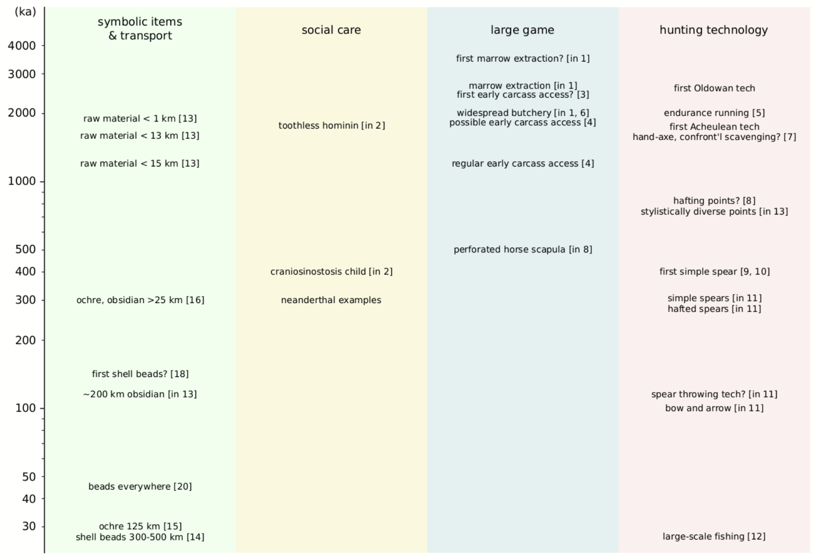
Two main known results:

1. Cooperation can be sustained
   - Do people 'mistake' linear games for a nonlinear ones?
2. But cooperation cannot invade
   - Imagine a small nbr. of cooperators invading defectors...



But what if, instead of randomly formed groups, groups tend to form with family members? Then invading Cooperators more likely to be grouped with other Cooperators.

Claim: Genetic homophily was higher in the past



Replicator dynamics with homophilic group formation

expected payoff to x-strategists

$$\dot{p}_x = p_x \left(\bar{\pi}_x - \sum_{i=1}^{m} p_i \bar{\pi}_i \right)$$

proportion of x-strategists

expected payoff in population

but now expected payoff:

no longer binomial

$$\bar{\pi}_C = \sum_{g_{\mathrm{nf}}=0}^{n-1} \pi(e_C, g_{\mathrm{nf}}) \; \mathbb{P}[G_{\mathrm{nf}} = g_{\mathrm{nf}} \mid G_0 = e_C]$$

nonfocal strategy distribution depends on focal's strategy

Colours are strategies, boxes are families: 

Some notation (Hisashi's previous work)

Let ρ_ℓ be the probability that ℓ players sampled without replacement from the group have strategy 1.

$$\rho_\ell = \sum_{m=1}^{\ell} \overbrace{\theta_{\ell \to m}}^{\text{prob. } \ell \text{ sampled have } m \text{ common ancestors}} \; \underbrace{p_1}_{\text{propn. strategy-1 in populatn}}{}^{m}$$

Examples:

- Sample 1 individual: $\rho_1 = p_1$
- Sample 2 individuals:

$$\rho_2 = \overbrace{\theta_{2 \to 1}}^{\text{prob. same ancestor}} \; \overbrace{p_1}^{\text{prob. strategy-1}} + \underbrace{\theta_{2 \to 2}}_{\text{prob. two ancestors}} \; \underbrace{p_1^2}_{\text{prob. both strategy-1}}$$

Hisashi's equation

Ohtsuki (2014, Phil Trans R Soc):

$$\dot{p}_1 = \sum_{g_{\mathrm{nf}}=0}^{n-1} \sum_{\ell=g_{\mathrm{nf}}}^{n-1} (-1)^{\ell-g_{\mathrm{nf}}} \binom{\ell}{g_{\mathrm{nf}}} \binom{n-1}{\ell}$$

relatedness terms

$$\Big[(1-\rho_1)\,\rho_{\ell+1}\,\pi(e_1, g_{\mathrm{nf}}) - \rho_1\,(\rho_\ell - \rho_{\ell+1})\,\pi(e_2, g_{\mathrm{nf}})\Big]$$

payoff terms

Linear PGG is a function of dyadic relatedness only

- If the PGG is linear, only need dyadic relatedness

$$\dot{p}_1 = f(\underbrace{\theta_{2\to 1}}_{\text{dyadic relatedness, Hamilton's } r})$$

because:

- Payoff function in n-player linear game can be written as a sum of payoffs in 2-player games

$$\underbrace{\pi^{(n)}(e_x, g_{\text{nf}}^{(n)})}_{\text{payoff in } n\text{-player game}} \equiv \sum_{g_{\text{nf}}^{(2)}} \underbrace{\pi^{(2)}(e_x, g_{\text{nf}}^{(2)})}_{\text{payoff in 2-player game}}$$

- So n-player linear game = sum of 2-player games
- So only dyadic relatedness is needed to calculate expected payoff

- But if the payoff function is nonlinear, higher-order relatedness coefficients are needed (e.g., $\theta_{3\to 1}, \theta_{3\to 2}, \theta_{4\to 1}$, etc.)

How do we calculate the higher-order relatedness terms?

From group family-size distribution. For example:

	partition	$\theta_{2\to1}$	explanation
$F_{[4]}$		1	Any 2 will have a common ancestor.
$F_{[3,1]}$		$\frac{3}{4} \times \frac{2}{3} = \frac{1}{2}$	Both must be blue (family size 3).
$F_{[2,2]}$		$1 \times \frac{1}{3} = \frac{1}{3}$	Choose any, then its 1 family member.
$F_{[2,1,1]}$		$\frac{2}{4} \times \frac{1}{3} = \frac{1}{6}$	Only possible in the partition of 2.
$F_{[1,1,1,1]}$		0	Not possible.

So if we can calculate the $F_{\boldsymbol{q}}$, we can calculate the needed $\theta_{l\to m}$

Homophilic group-formation models

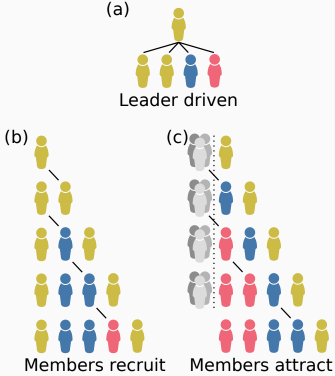
Kristensen *et al.* (2022); Martin & Lessard

(a) Leader driven:

- The leader is chosen at random from the population.
- Leader recruits/attracts kin with probability h and nonkin with probability $1 - h$.
- Group family size distribution

$$F_{[\ell,1,\ldots,1]} = \binom{n-1}{\ell-1} h^{\ell-1}(1-h)^{n-\ell}.$$

$h =:$ genetic homophily



Homophilic group-formation models

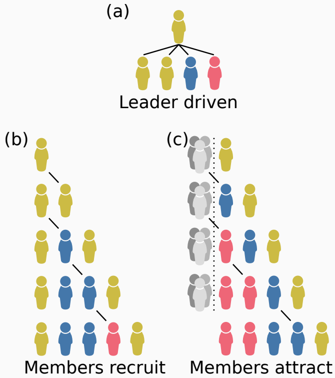
(b) Members recruit:

- All group members have an equal chance to recruit the next member.
- Equation in Kristensen *et al.* (2022)

(c) Members attract:

- Outsiders attracted to kin
- But also attracted to the group as a whole
- Use Ewens' formula (Ewen 1972).

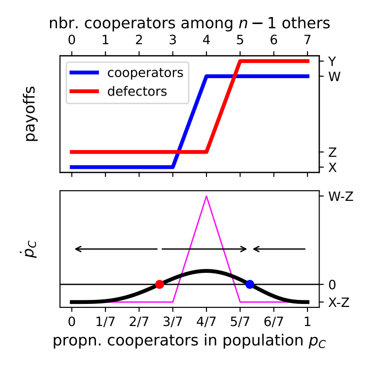
$h =:$ genetic homophily

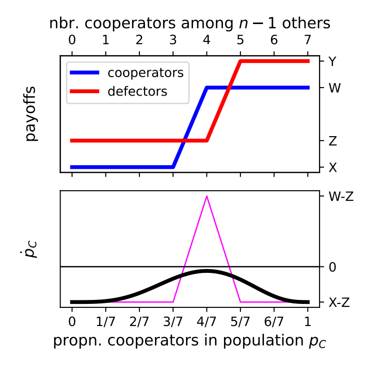


NOTE: can be interpreted in terms of 'matching rules', i.e., strategy homophily *sensu* Jensen & Rigos (2018, Int J Game Theory)

Results

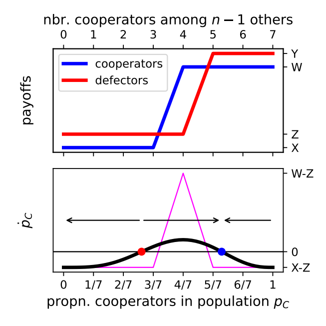
Recall no-homophily result: cooperation can (sometimes) persist but it can never invade:



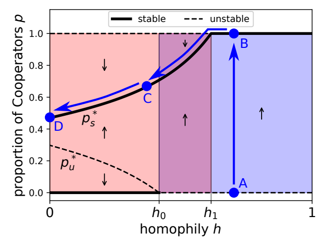


We want to go backwards in time — increase homophily — and see if cooperation can invade.

Results

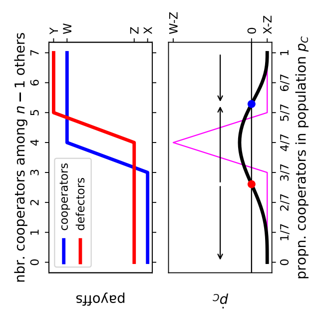


nbr. cooperators among $n-1$ others
0 1 2 3 4 5 6 7
cooperators
defectors
payoffs
Y
W
Z
X
W-Z
0
X-Z
$\dot{p}_C$
0 1/7 2/7 3/7 4/7 5/7 6/7 1
propn. cooperators in population p_C

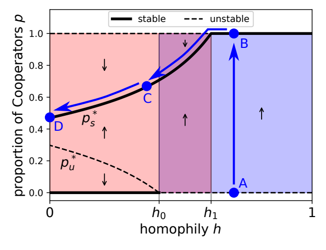


stable
unstable
proportion of Cooperators p
1.0
0.8
0.6
0.4
0.2
0.0
B
C
D
A
p_s^*
p_u^*
0
h_0
h_1
1
homophily h

Results



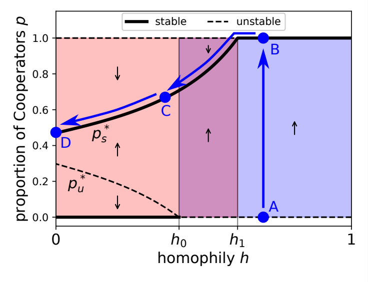
nbr. cooperators among n − 1 others
payoffs
cooperators
defectors
propn. cooperators in population p_C
$\dot{p}_C$

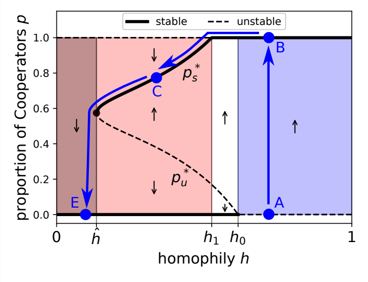


stable
unstable
proportion of Cooperators p
homophily h
p_s^*
p_u^*
h_0
h_1
A
B
C
D

Results

- Cooperation cannot invade a threshold game
    - Also true for sigmoid games in general (Peña *et al.*, 2014)
- Can arise through historical homophily





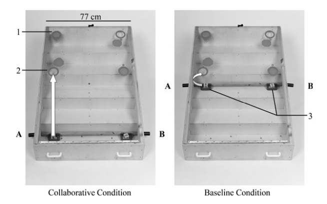
- For cooperation to persist, either:
    - Parameters such that it can be sustained in a well-mixed population
    - Some degree of homophily maintained

Many discrete strategies

- So far, 2 strategies; natural extension, *m* strategies
- Discrete strategies:
  - I could have modelled cooperate and defect as *degree* of cooperation — one continuous strategy
  - However, some strategies are naturally discrete
    - e.g., conditioning on the actions of others
  - Shared intentionality (Genty et al., 2020; Tomasello, 2020):
    - form a collective 'we' with a jointly optimised goal
    - make a joint commitment (!?) to the goal
    - coordinate our actions towards achieving it

Commitment

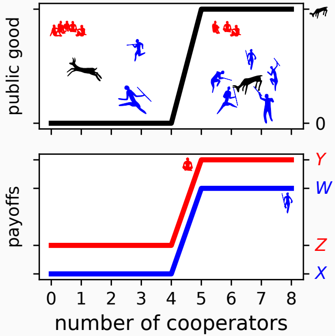
- Commitment is a norm: one *should* do what one promised
  - Kerr and Kaufman-Gilliland (1994, J Pers Soc Psychol)
- Commitment distinguishes us from other apes
  - In a experimental situation where one individual receives their reward early, 3.5-year-old children will continue contributing until their partner also receives their reward (Hamann et al., 2012), whereas chimpanzees don't distinguish between continuing to help in an existing collaboration versus starting a new one (Greenberg et al., 2010).



Hamann and Warneken (2012, Child Dev)

Commitment and coordination

- In the threshold game, hunters are a bit stupid
  - Cooperator will run off to do the hunt by themselves
- But people don't really behave this way – they coordinate
  - If we were in this situation, we'd have a conversation
  - And that's also how people behave experimentally (e.g., Van de Kragt *et al.* 1983, Am Pol Sci Rev)
- Plus, coordination improves the evolutionary prospects for cooperation!



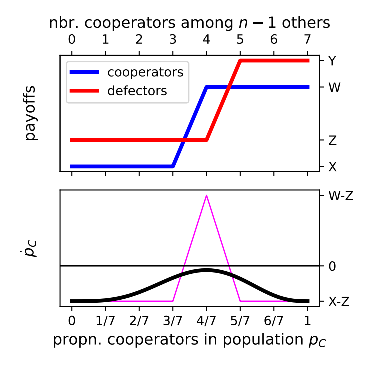
- Newton (2017 Games Econ Behav) 'shared intentionality' evolves under fairly general conditions in a public goods game

Jonathan Newton

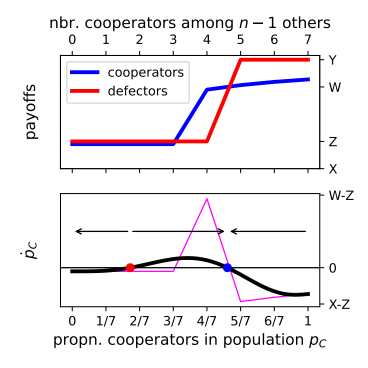
Coordination in a threshold game example

- Extend the threshold game:
  - Coordinating cooperators draw straws to decide who will contribute
  - The ability to coordinate entails a small cognitive cost ε

old threshold game

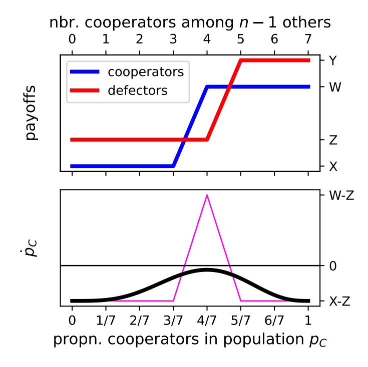


coordinated cooperation game

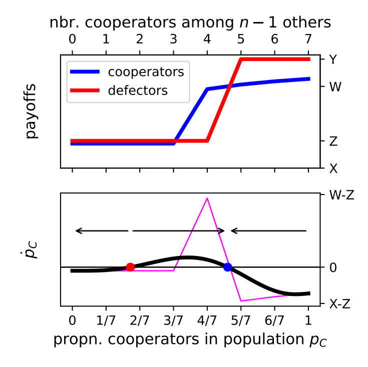


Coordination in a threshold game example

old threshold game



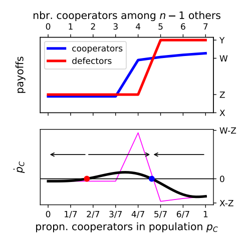
coordinated cooperation game



- Sustains cooperation where it could not otherwise be sustained
- Can't invade, but we already know we can overcome this with homophily

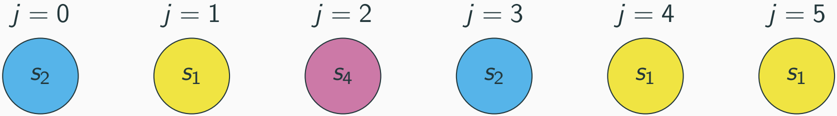
Even in a linear game?

- Coordination can even sustain cooperation in a linear game! ... wait
  - It never makes sense to contribute in the linear game
  - It's true the Defectors can't invade, but what about a type who participates in the lottery but doesn't follow through?
- New strategy: Liars



New notation

- $\boldsymbol{G}$ random variable for strategy composition, takes values $\boldsymbol{g}$
- Subscripts: 0 = focal player; nf = nonfocal players; a = all players



- Players: $\boldsymbol{g}_0 = (0, 1, 0, 0)$, $\boldsymbol{g}_1 = (1, 0, 0, 0)$, $\boldsymbol{g}_2 = (0, 0, 0, 1), \ldots$
- Whole-group: $\boldsymbol{g}_{\mathrm{a}} = (3, 2, 0, 1)$
- Nonfocal: $\boldsymbol{g}_{\mathrm{nf}} = (3, 1, 0, 1)$
- $\boldsymbol{g}_j = \boldsymbol{e}_x$: player j plays strategy s_x (a 1 in the x-th position)

Many strategies

How does a trait change frequency over time?

dynamics of propn. of s_x

$$\Delta p_x = \mathbb{C}\mathrm{ov}[\, G_{0,x} \,, W_0 \,],$$

focal's strategy indicator

fitness of focal

$$G_{0,x} = \begin{cases} 1 & \text{if focal strategy } s_x, \\ 0 & \text{otherwise.} \end{cases}$$

(... *some useful covariance identities* ...)

strategy indicator

focal payoff

$$\Delta p_x \propto \mathbb{C}\mathrm{ov}\left[\, G_{0,x} \,, \Pi_0 \right]$$

George Robert Price

Other member accounting

$$\Delta p_x \propto \mathbb{C}\mathrm{ov}\left[\overbrace{G_{0,x}}^{\text{focal's strategy indicator}}, \overbrace{\Pi_0}^{\text{focal payoff}}\right]$$

Payoff to the focal individual:

$$\Pi_0 = \sum_{i=1}^{m} \overbrace{G_{0,i}}^{\text{1 if focal plays } s_i\text{; 0 otherwise}} \overbrace{\pi}^{\text{payoff to } s_i\text{-player}}(\boldsymbol{e}_i, \underbrace{\boldsymbol{G}_{\mathrm{nf}}}_{\text{nonfocal strategy composition}})$$

Useful identity: $\mathbb{C}\mathrm{ov}[X, Y] = \mathbb{E}[XY] - \mathbb{E}[X]\,\mathbb{E}[Y]$

$$\Delta p_x = \mathbb{E}\left[G_{0,x}\,\pi(\boldsymbol{e}_x, \boldsymbol{G}_{\mathrm{nf}})\right] - p_x \sum_{i=1}^{m} \mathbb{E}\left[G_{0,i}\,\pi(\boldsymbol{e}_i, \boldsymbol{G}_{\mathrm{nf}})\right]$$

nonfocal strategy composition

Other member accounting

nonfocal strategy composition

$$\Delta p_x = \mathbb{E}\left[G_{0,x}\,\pi(\boldsymbol{e}_x, \boldsymbol{G}_{\mathrm{nf}})\right] - p_x \sum_{i=1}^{m} \mathbb{E}\left[G_{0,i}\,\pi(\boldsymbol{e}_i, \boldsymbol{G}_{\mathrm{nf}})\right]$$

Let $\mathcal{G}_{\mathrm{nf}}$ be the set of all strat. compositions $\boldsymbol{g}_{\mathrm{nf}}$. Then expectations:

$$\begin{aligned}
\mathbb{E}\left[G_{0,i}\pi(\boldsymbol{e}_i, \boldsymbol{G}_{\mathrm{nf}})\right] &= \sum_{\boldsymbol{g}_{\mathrm{nf}} \in \mathcal{G}_{\mathrm{nf}}} \pi(\boldsymbol{e}_i, \boldsymbol{g}_{\mathrm{nf}})\, \mathbb{P}[\boldsymbol{G}_0 = \boldsymbol{e}_i, \boldsymbol{G}_{\mathrm{nf}} = \boldsymbol{g}_{\mathrm{nf}}] \\
&= \sum_{\boldsymbol{g}_{\mathrm{nf}} \in \mathcal{G}_{\mathrm{nf}}} \pi(\boldsymbol{e}_i, \boldsymbol{g}_{\mathrm{nf}})\, \underbrace{\mathbb{P}[\boldsymbol{G}_0 = \boldsymbol{e}_i]}_{p_i}\, \mathbb{P}[\boldsymbol{G}_{\mathrm{nf}} = \boldsymbol{g}_{\mathrm{nf}} \mid \boldsymbol{G}_0 = \boldsymbol{e}_i] \\
&= p_i \underbrace{\sum_{\boldsymbol{g}_{\mathrm{nf}} \in \mathcal{G}_{\mathrm{nf}}} \pi(\boldsymbol{e}_i, \boldsymbol{g}_{\mathrm{nf}})\, \mathbb{P}[\boldsymbol{G}_{\mathrm{nf}} = \boldsymbol{g}_{\mathrm{nf}} \mid \boldsymbol{G}_0 = \boldsymbol{e}_i]}_{\overline{\pi}_i}
\end{aligned}$$

Recovered replicator eqn: $\Delta p_x \propto p_x \left(\overline{\pi}_x - \sum_{i=1}^{m} p_i \overline{\pi}_i\right) = p_x \left(\overline{\pi}_x - \overline{\pi}\right)$.

But $\mathbb{P}[\boldsymbol{G}_{\mathrm{nf}} = \boldsymbol{g}_{\mathrm{nf}} \mid \boldsymbol{G}_0 = \boldsymbol{e}_i]$ is not obvious: 

Whole-group accounting

Idea: draw a group at random, then draw a focal individual.

$$\Delta p_x \propto \mathbb{C}\mathrm{ov}\left[\underbrace{G_{0,x}}_{\text{strategy indicator}}, \underbrace{\Pi_0}_{\text{focal payoff}}\right]$$

This time, focus on the whole-group distribution.

new payoff fnc wrt whole-group strategy composition

$$\Pi_0 = \sum_{i=1}^{m} G_{0,i}\ \hat{\pi}(\boldsymbol{e}_i, \boldsymbol{G}_{\mathrm{a}})$$

Using a similar method to before involving covariance identities and re-arranging, we obtain

$$\Delta p_x = \sum_{\boldsymbol{g}_{\mathrm{a}} \in \mathcal{G}_{\mathrm{a}}} \left(\frac{g_{\mathrm{a},x}}{n} \hat{\pi}(\boldsymbol{e}_x, \boldsymbol{g}_{\mathrm{a}}) - p_x \sum_{i=1}^{m} \frac{g_{\mathrm{a},i}}{n} \hat{\pi}(\boldsymbol{e}_i, \boldsymbol{g}_{\mathrm{a}}) \right) \mathbb{P}[\boldsymbol{G}_{\mathrm{a}} = \boldsymbol{g}_{\mathrm{a}}]$$

prob. of whole-group strategy composition

Prob. of whole-group strategy composition, $\mathbb{P}[\boldsymbol{G}_{\mathbf{a}} = \boldsymbol{g}_{\mathbf{a}}] =?$

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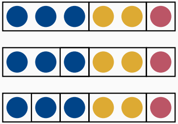
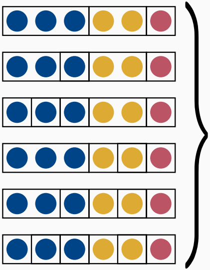


Prob. of whole-group strategy composition, $\mathbb{P}[G_{\mathrm{a}} = g_{\mathrm{a}}] = ?$

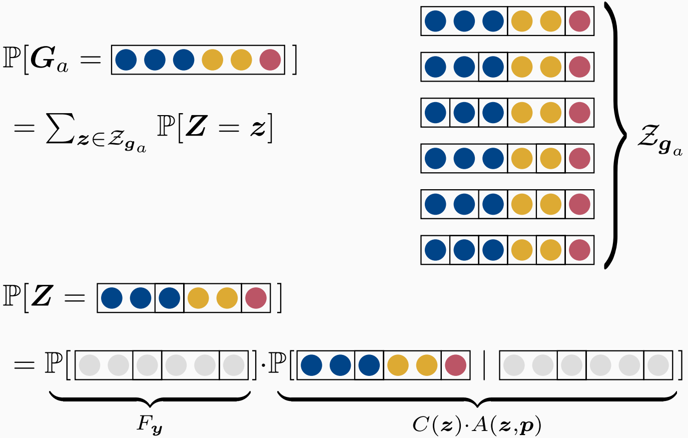
$\mathbb{P}[\boldsymbol{G}_a =$  $]$

$= \sum_{\boldsymbol{z} \in \mathcal{Z}_{\boldsymbol{g}_a}} \mathbb{P}[\boldsymbol{Z} = \boldsymbol{z}]$



$\mathcal{Z}_{\boldsymbol{g}_a}$

Prob. of whole-group strategy composition, $\mathbb{P}[G_{\mathrm{a}} = g_{\mathrm{a}}] = ?$



Probability of whole-group strategy composition

Probability of strategywise family-size distribution:

get from homophilic group-formation model

$$\mathbb{P}[\boldsymbol{G}_{\mathrm{a}} = \boldsymbol{g}_{\mathrm{a}}] = \sum_{\boldsymbol{z} \in \mathcal{Z}_{\boldsymbol{g}_{\mathrm{a}}}} F_{\boldsymbol{y}} \; C(\boldsymbol{z}) \; A(\boldsymbol{z}, \boldsymbol{p})$$

count of multiset permutations

prob. families' strategies

nbr. families pursuing strategy s_i

$$A(\boldsymbol{z}, \boldsymbol{p}) = \prod_{i=1}^{m} p_i^{\|\boldsymbol{z}_i\|}$$

Analogous to the power terms in 2-strategy game, e.g.,

$$\rho_2 = \theta_{2\to 1} \; p_1 + \theta_{2\to 2} \; p_1^2$$

Whole-group accounting

Bringing it all together:

$$\Delta p_x \propto \underbrace{\sum_{\boldsymbol{g}_\mathrm{a} \in \mathcal{G}_\mathrm{a}}}_{\text{sum over group strategy compositions}} \left(\overbrace{\frac{g_{\mathrm{a},x}}{n}}^{\text{prob. focal pursues } s_x} \hat{\pi}(\boldsymbol{e}_x, \boldsymbol{g}_\mathrm{a}) - p_x \sum_{i=1}^{m} \frac{g_{\mathrm{a},i}}{n} \hat{\pi}(\boldsymbol{e}_i, \boldsymbol{g}_\mathrm{a}) \right) \left(\overbrace{\sum_{\boldsymbol{z} \in \mathcal{Z}_{\boldsymbol{g}_\mathrm{a}}}}^{\text{over strategywise family-sizes}} C(\boldsymbol{z})\, A(\boldsymbol{z}, \boldsymbol{p})\, F_{\mathrm{sum}(\boldsymbol{z})} \right)$$

- Not as intuitive as the traditional replicator equation
  - $\Delta p_x \propto p_x \left(\overline{\pi}_x - \overline{\pi} \right)$
- Might be useful from computational perspective because we've split homophily calculations off from strategy identity
- Now it's clearer how to calculate $\mathbb{P}[\boldsymbol{G}_\mathrm{nf} = \boldsymbol{g}_\mathrm{nf} \mid \boldsymbol{G}_0 = \boldsymbol{e}_i]$

Aside: Payoff-matrix transformation example (2 players)

- Idea: transform payoffs so they take into account homophily
- Well-mixed game: $\dot{p}_i = p_i(\overline{\pi}_i - \overline{\pi}) = p_i((A\,\boldsymbol{p})_i - \boldsymbol{p}^T A\,\boldsymbol{p})$, where $a_{i,j} = \pi(\boldsymbol{e}_i, \boldsymbol{e}_j)$,

$$\overline{\boldsymbol{\pi}} = \begin{pmatrix} \overline{\pi}_1 \\ \vdots \\ \overline{\pi}_m \end{pmatrix} = \begin{array}{c c} & \xrightarrow{\text{nonfocal's strategy}} \\ \text{focal's strat.}\downarrow & \begin{pmatrix} a_{1,1} & \dots & a_{1,m} \\ \vdots & & \vdots \\ a_{m,1} & \dots & a_{m,m} \end{pmatrix} \end{array} \begin{pmatrix} p_1 \\ \vdots \\ p_m \end{pmatrix} = \begin{pmatrix} a_{1,1}p_1 + \ldots + a_{1,m}p_m \\ \vdots \\ a_{m,1}p_1 + \ldots + a_{m,m}p_m \end{pmatrix}$$

- Now with homophily, dyadic relatedness $\theta_{2\to1}$

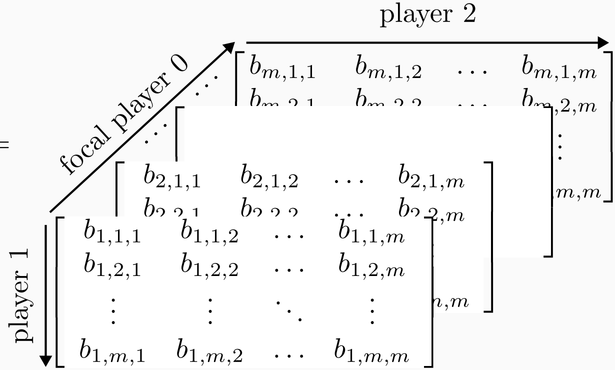
$$B = \underbrace{\theta_{2\to1} \begin{pmatrix} a_{1,1} & \dots & a_{1,1} \\ \vdots & & \vdots \\ a_{m,m} & \dots & a_{m,m} \end{pmatrix}}_{i \text{ matched with } i \text{ with prob. } \theta_{2\to1}} + \underbrace{(1-\theta_{2\to1}) \begin{pmatrix} a_{1,1} & \dots & a_{1,m} \\ \vdots & & \vdots \\ a_{m,1} & \dots & a_{m,m} \end{pmatrix}}_{i \text{ matched with random with prob. } 1-\theta_{2\to1}}$$

- Dynamics of A with homophily $\equiv$ dynamics of B well-mixed

$$\dot{p}_i = p_i((B\,\boldsymbol{p})_i - \boldsymbol{p}^T B\,\boldsymbol{p})$$

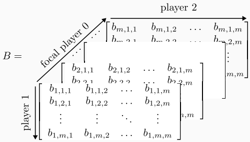
Seeking a solution to:

$B =$



$$B_1 = \begin{bmatrix} b_{1,1,1} & b_{1,1,2} & \dots & b_{1,1,m} \\ b_{1,2,1} & b_{1,2,2} & \dots & b_{1,2,m} \\ \vdots & \vdots & \ddots & \vdots \\ b_{1,m,1} & b_{1,m,2} & \dots & b_{1,m,m} \end{bmatrix}$$

Aside: Payoff transformation n players



$$b_{\boldsymbol{u}} = \sum_{\boldsymbol{q} \vdash n} F_{\boldsymbol{q}} \left(\sum_{q_0 \in \boldsymbol{q}} \frac{q_0}{n|\mathcal{J}_{q_0,\boldsymbol{q}}|} \left(\sum_{\boldsymbol{j} \in \mathcal{J}_{q_0,\boldsymbol{q}}} a_{\boldsymbol{u}_{\boldsymbol{j}}} \right) \right)$$

get from group-formation model

Code to calculate it on Github:

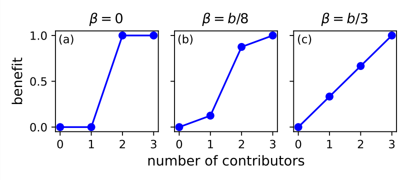
1. Numerically: `TransmatBase` class `functions/transmat_base.py`.
2. Symbolically: `functions/symbolic_transformed.py`.

But why would you want to do this?

- B is expensive to calculate, but matrix multiplication is optimised, can be worth the trade-off when finding steady states
- Use maths from well-mixed case, e.g., Jorge Peña's analysis techniques (example in appendix)

Coordinated cooperation

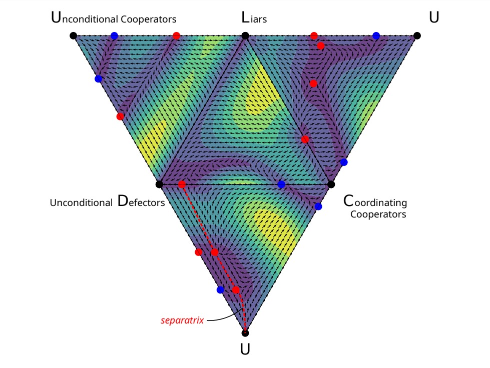
- Game with 4 strategies:
  1. D: unconditional Defector, never contributes
  2. C: Coordinating cooperator, hold lottery, follow through if chosen
     - Nbr. contributors $\tau =$ threshold, or inflection point if sigmoid
  3. L: Liar, participate in lottery, never contributes
  4. U: Unconditional cooperator, always contributes
- C and L pay cognitive cost ε regardless of game outcome
- U and C pay contribution cost c if contributing
- Explore the range from linear to threshold game



Example 3-player - symbolic analysis

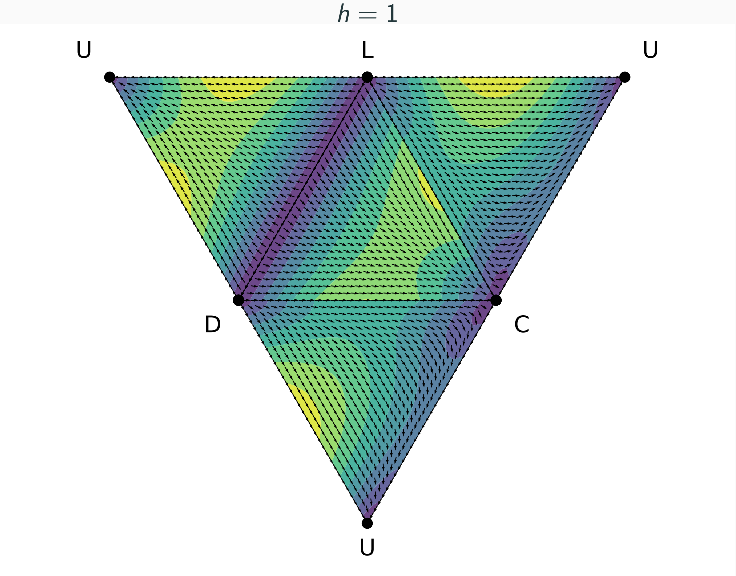
Example 8-player - numerical analysis

How to read results



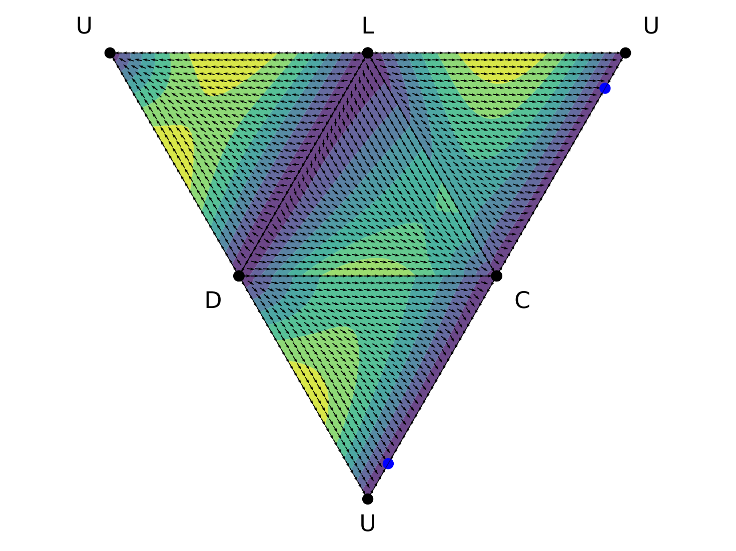
- Evolutionary dynamics for a given homophily level h
  - Dynamics inside a triangular pyramid
  - The points represent a population with just one strategy, lines 2 strategies, triangles 3
  - Blue points are stable in that dimension, red points unstable

Results 1: fairly nonlinear benefits function



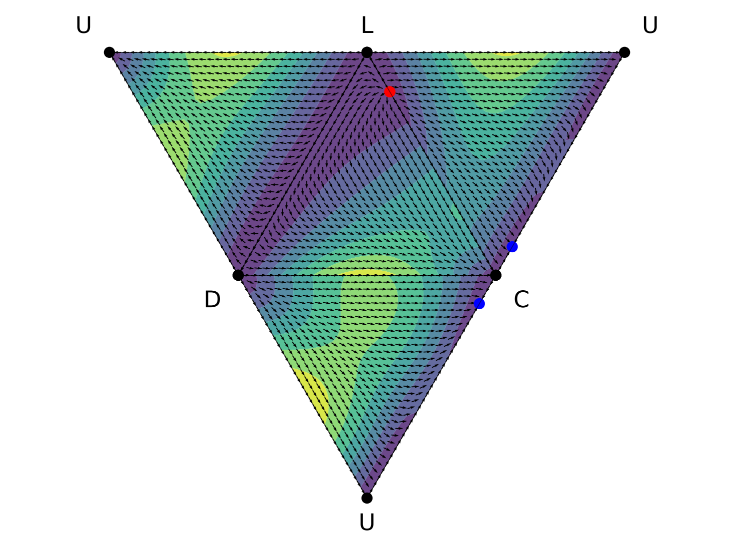
Results 1: fairly nonlinear benefits function

$h = 0.645$

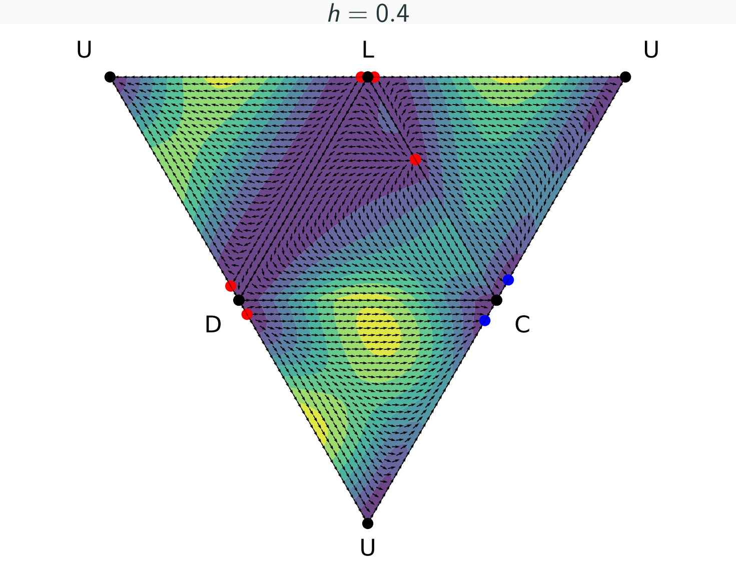


Results 1: fairly nonlinear benefits function

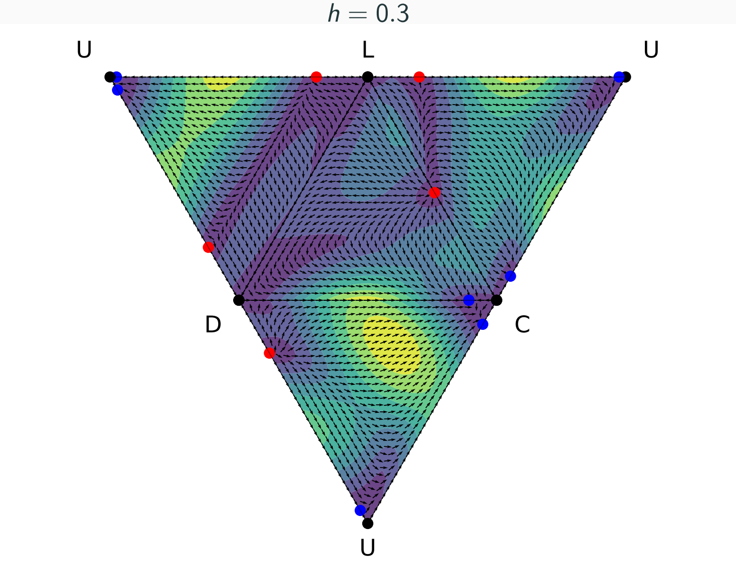
$h = 0.5$



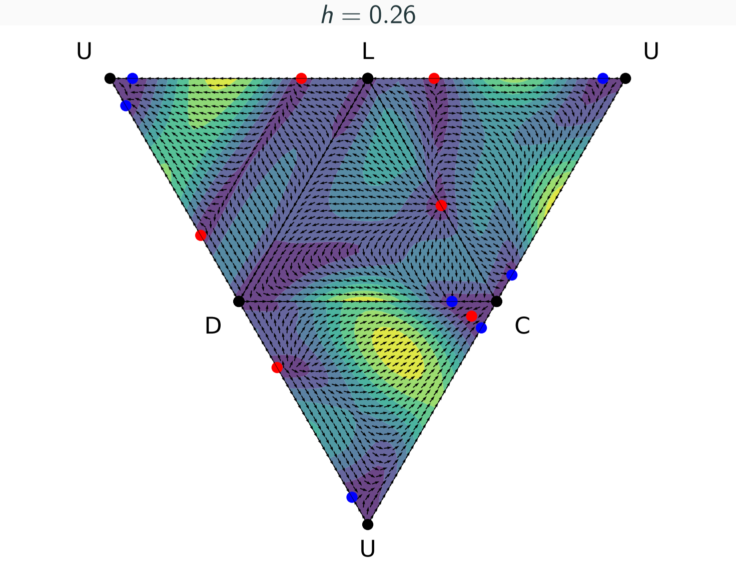
Results 1: fairly nonlinear benefits function



Results 1: fairly nonlinear benefits function

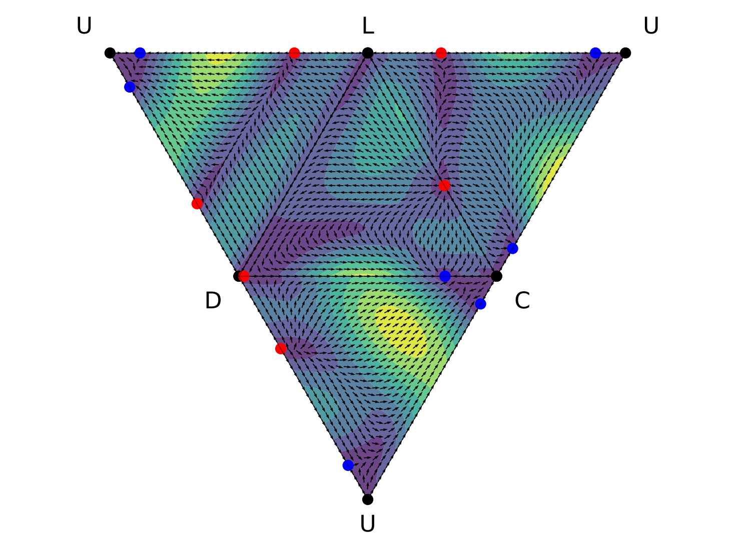


Results 1: fairly nonlinear benefits function



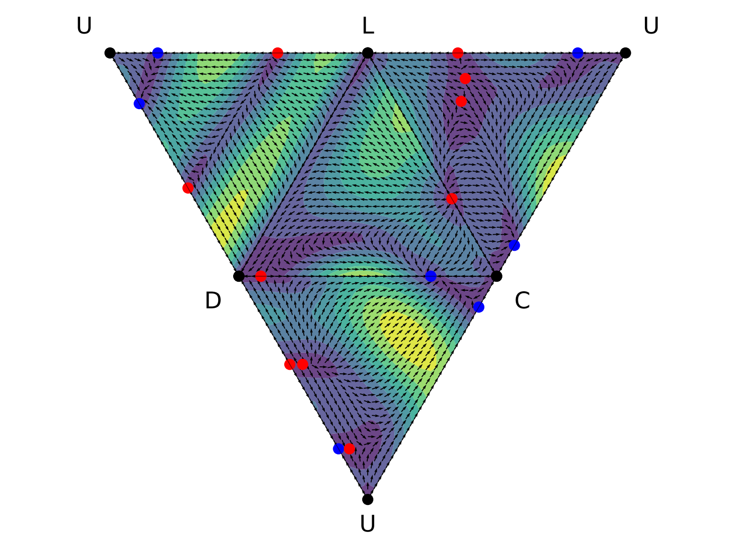
Results 1: fairly nonlinear benefits function

$h = 0.24$

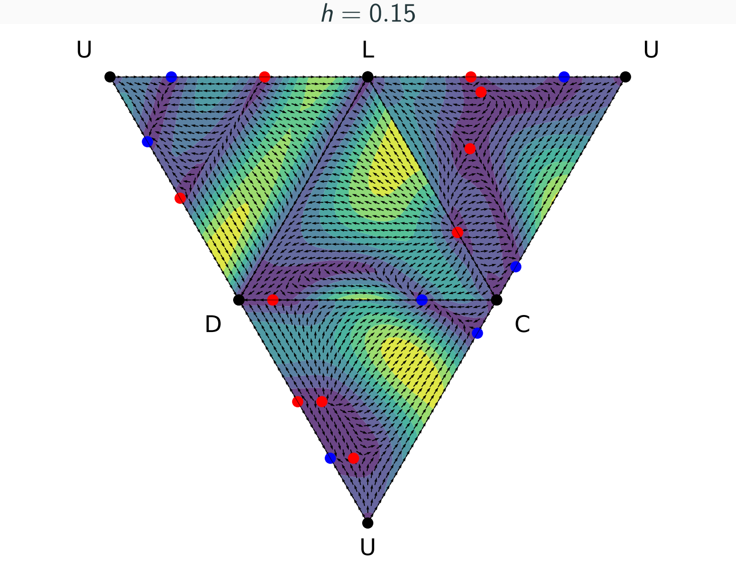


Results 1: fairly nonlinear benefits function

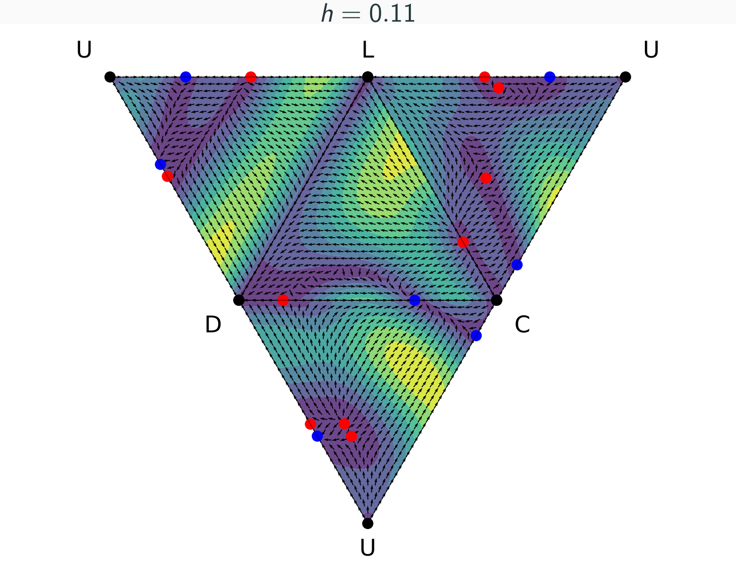
$h = 0.19$



Results 1: fairly nonlinear benefits function

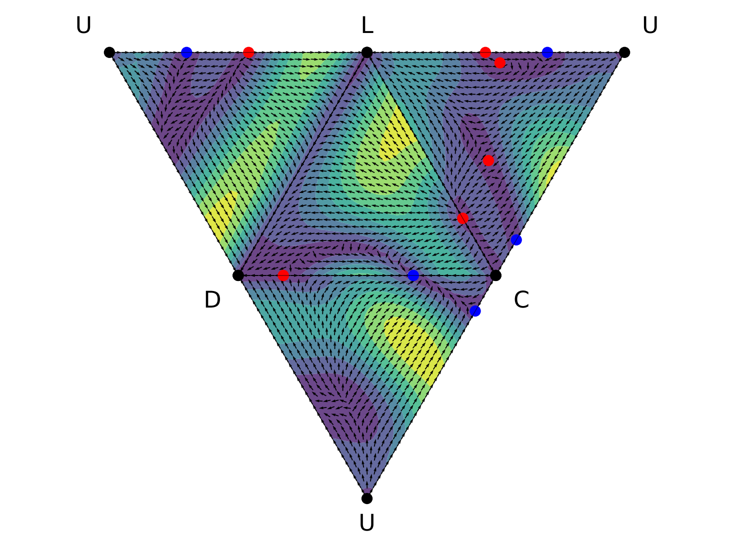


Results 1: fairly nonlinear benefits function



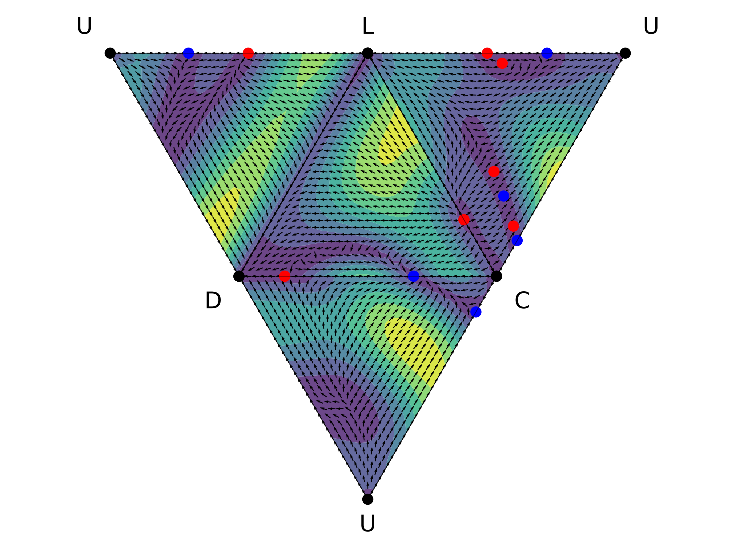
Results 1: fairly nonlinear benefits function

$h = 0.106$

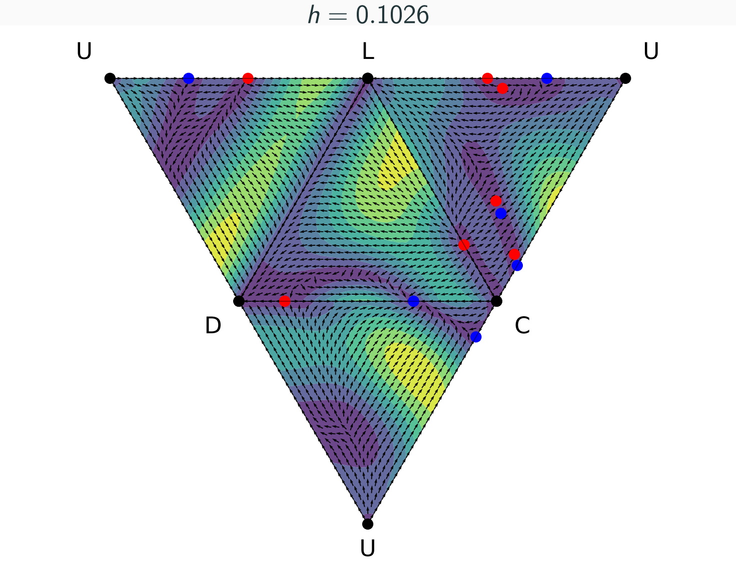


Results 1: fairly nonlinear benefits function

$h = 0.1031$

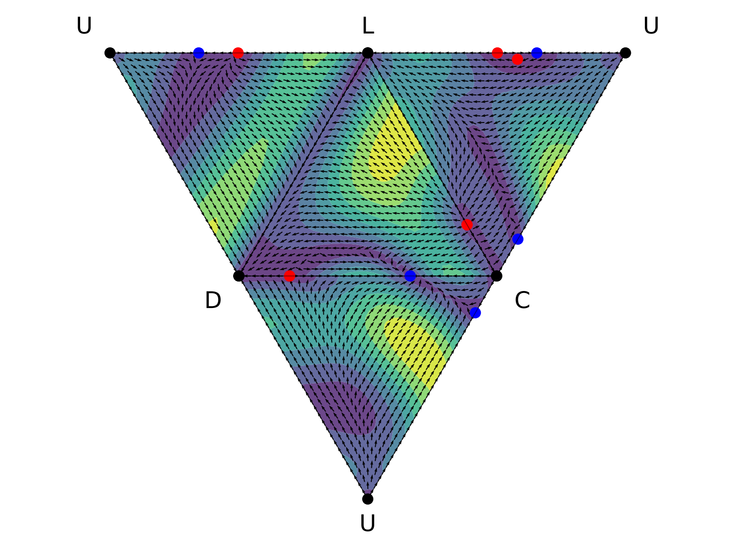


Results 1: fairly nonlinear benefits function



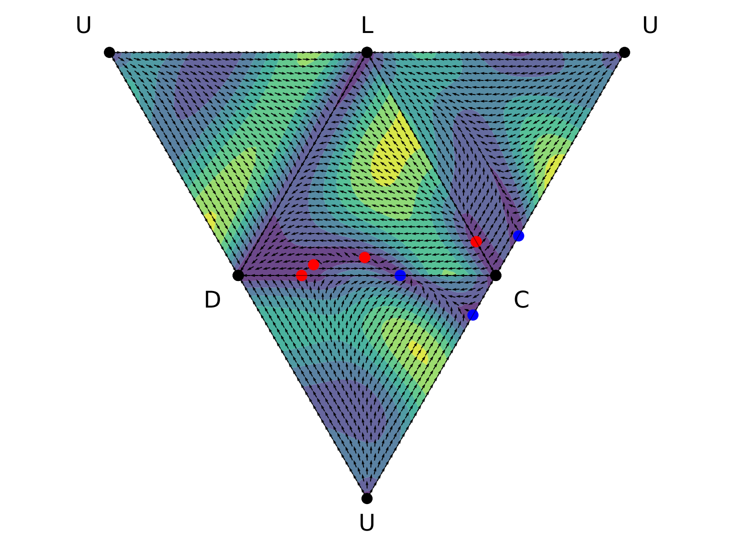
Results 1: fairly nonlinear benefits function

$h = 0.08$



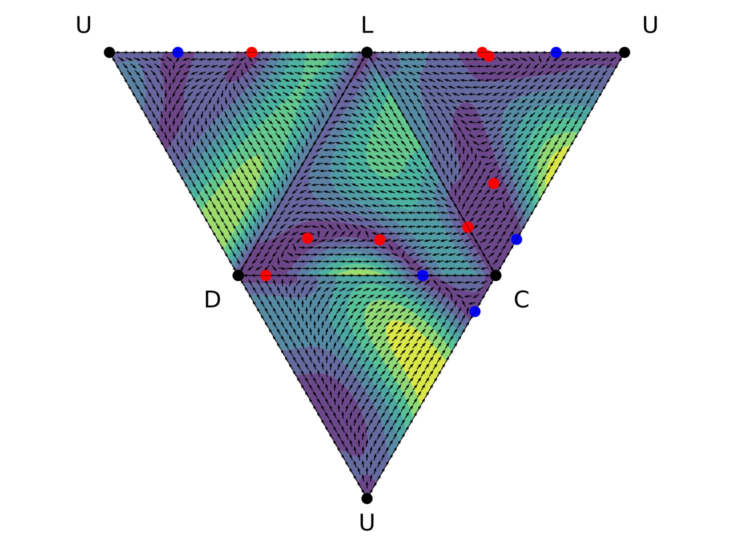
Results 1: fairly nonlinear benefits function

$h = 0$

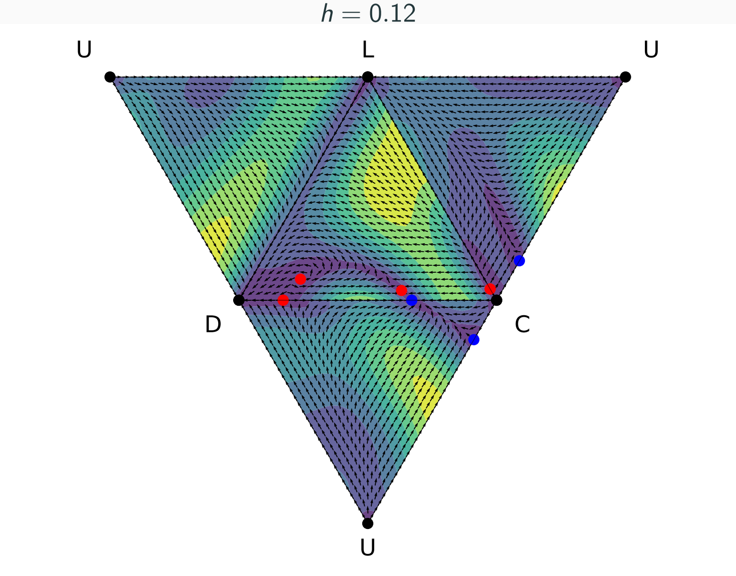


Results 2: more linear benefits function

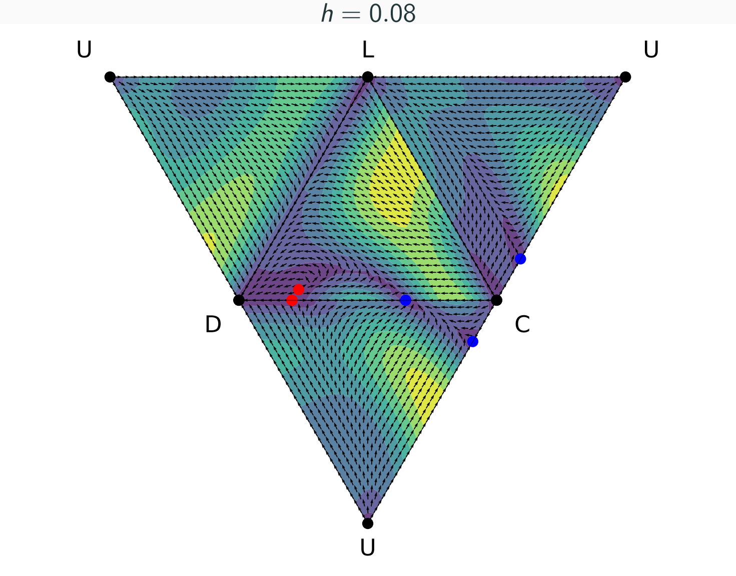
$h = 0.18$



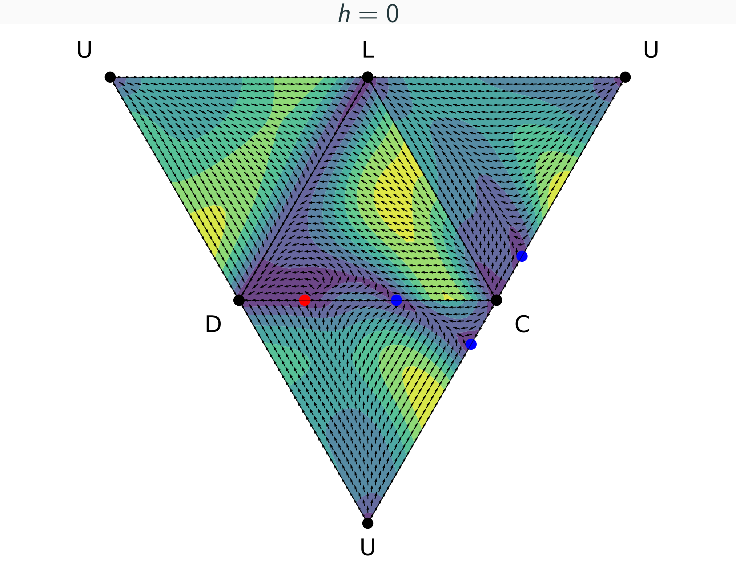
Results 2: more linear benefits function



Results 2: more linear benefits function



Results 2: more linear benefits function



Summary

- Mathematical framework combines discrete-strategy group games with kin selection (or ‘matching rules’)
- Investigate how cooperation first arose and how it can persist

github.com/nadiahpk

nadiah.org