1.2 Working through the Supplementary of Padget et al.

Starting in Supplementary Section 6 of Padget et al. (2023), I get the same result as the first paragraph:

> ### 6 Simple examples of changes to the volunteer's dilemma that impact the effect of group size
>
> #### 6.1 Example demonstrating impact when group size modulates cost of volunteering
>
> Original volunteer's dilemma (as in Archetti, 2009a) gives the payoff from volunteering (W_v), payoff from ignoring (W_i) and the probability of ignoring at equilibrium (γ_{eq}; when $W_v = W_i$):
>
> $$W_v = 1 - c \tag{1}$$
>
> $$W_i = \gamma^{(N-1)}(1-a) + (1-\gamma^{(N-1)}) \tag{2}$$
>
> $$\gamma_{eq} = \left(\frac{c}{a}\right)^{\frac{1}{N-1}}. \tag{3}$$
>
> Using the examples in Archetti (2009a), when $c = 0.3$ and $a = 1$: $N = 2$ gives the probability that everyone ignores ($\gamma_{eq}^{1/(N-1)}$) as 0.09; $N = 10$ gives 0.26. The probability of ignoring increases, thus the probability of volunteering decreases as group size increases.

From Archetti (2009), the payoff for being a volunteer is

$$W_v = 1 - c$$

and for ignoring is

$$W_i = \underbrace{\gamma^{N-1}(1-a)}_{\text{no one volunteers}} + \underbrace{1-\gamma^{N-1}}_{\text{at least 1 volunteer}}$$

where γ is the probability of ignoring. When no one volunteers, everyone pays a cost a, and if at least one volunteers, ignorers get the benefit 1. At the mixed-strategy equilibrium, the payoffs to each strategy are equal, which results in the expression

$$\gamma_{\text{eq}} = \left(\frac{c}{a}\right)^{\frac{1}{N-1}}.$$

Because $c/a < 1$ (default values $c = 0.3$ and $a = 1$), the probability of ignoring γ_{eq} increases as N increases.

A key result from Archetti (2009) is that the probability that no one volunteers is proportional to the ignoring probability

$$\mathbb{P}[\text{no one volunteers}] = \gamma_{\text{eq}}^N = \gamma_{\text{eq}}\frac{c}{a},$$

(not $\gamma_{\text{eq}}^{1/(N-1)}$ above). Therefore, because γ_{eq} increases with increasing N, the probability that no one volunteers increases with N.

I can verify that I get the same results for $N = 2$ and $N = 10$ (in green above)

```
# parameter values, defaults from Archetti
c = 0.3
a = 1

# Archetti's function
gamma_eq_fnc = lambda N: (c/a)**(1/(N-1))
```

```
# everyone ignores
ignore_fnc = lambda N: gamma_eq_fnc(N)**N

NV = [2, 10]

ignoreV = [ignore_fnc(N) for N in NV]
```

Result matches probabilities they give:

```
ignoreV
[0.09, 0.2624361248771798]
```

For the paragraph immediately following, I get different results.

> probability of volunteering decreases as group size increases.
>
> If we instead imagine that the cost (c) depends on the group size (N) such that:
>
> $$W_v = 1 - \frac{c}{N}, \tag{4}$$
>
> then,
>
> $$\gamma_{eq} = \frac{c}{aN}^{\frac{1}{N-1}}. \tag{5}$$
>
> The probability of ignoring when c = 0.3 and a = 1 is therefore 0.15 when N = 2 and 0.03 when N = 10. The probability of volunteering increases with group size.

```
# function for gamma_eq with N
gamma_eq_cd_fnc = lambda N: (c / (a * N))**(1 / (N-1))
```

Result:

```
[gamma_eq_cd_fnc(N) for N in [2, 10]]
[0.15, 0.6773158683865648]
```

Why is my result different? One possibility is that they meant 'probability everyone ignores' rather than 'probability of ignoring', which would match the calculation of interest in the previous paragraph. However, that would give

$$\mathbb{P}[\text{no one volunteers}] = \gamma_{\text{eq}} \frac{c}{aN},$$

and this does not give the result they quoted:

```
# probability everyone ignores
[gamma_eq_cd_fnc(N)**N for N in NV]
[0.0225, 0.020319476051596935]
```

Further, the probability everyone ignores is nonmonotonic with N, and initially increases and then decreases with N:

```
[gamma_eq_cd_fnc(N)**N for N in [2, 3, 4, 5]]
[0.0225, 0.0316227766016838, 0.031628724948815606, 0.029695392023038586]
```

With some trial and error, I noticed their probabilities match $\frac{c}{aN}$:

```
[c / (a*N) for N in [2, 10]]
[0.15, 0.03]
```

The next paragraph:

> The probability of volunteering increases with group size.
>
> It is probably more biologically plausible however that the cost of volunteering is modulated not by the whole group size but by the number of other volunteers, such that:
>
> $$W_v = 1 - \frac{c}{N_v}, \tag{6}$$
>
> then,
>
> $$\gamma_{eq} = \frac{c}{aN}^{\frac{1}{N-1}}, \tag{7}$$
>
> where N_v is the number of volunteers, as in Weesie and Franzen (1998). However, because the number of volunteers is still decreasing with group size, cost-sharing results in the same negative relationship between group size and volunteering.

Weesie and Franzen (1998) found the mixed-strategy equilibrium by finding $\gamma = \gamma_{\text{eq}}$ that solves

$$g(\gamma) = \gamma^{N-1}\left[c\gamma + aN(1-\gamma)\right] - c = 0.$$

The relationship between the equilibrium and N and the other parameters is found by implicit differentiation (appendix of Weesie and Franzen (1998); the same method I used in my blog post).

The next paragraph:

> relationship between group size and volunteering.
>
> 6.2 Example demonstrating impact when group size increases benefit of volunteering.
>
> If we implement a simple, linear effect of group size such that the benefit is given by the group size, we get the following:
>
> $$W_v = N - c, \tag{8}$$
>
> $$W_i = \gamma^{(N-1)}(1-a) + (1-\gamma^{(N-1)}) \tag{9}$$
>
> $$\gamma_{eq} = \left(\frac{-c}{1-a-N}\right)^{\frac{1}{N-1}}. \tag{10}$$
>
> However, γ_{eq} has no positive solutions for any integer value of N.

If I treat Eq. 9 as true, then for Eq. 10, I get

$$\gamma_{\text{eq}} = \left(\frac{1+c-N}{a}\right)^{\frac{1}{N-1}}.$$

As they say, does not have positive solutions.

Alternatively, I could replace Eq. 9 with

$$W_i = \gamma^{N-1}(1-a) + N(1-\gamma^{N-1}),$$

which gives their Eq. 10, but this does have positive solutions.

The next and final paragraph:

> If we want to allow group size to increase the benefit from volunteering non-linearly, we need a reward term that is not 1 (because 1^x is 1). We can introduce a term for the reward, r, to get:
>
> $$W_v = r^N - c, \tag{11}$$
>
> $$, \quad W_i = \gamma^{(N-1)}(1-a) + (1-\gamma^{(N-1)}) \tag{12}$$
>
> $$\gamma_{eq} = \left(\frac{-c}{1-a-r^N}\right)^{\frac{1}{N-1}}. \tag{13}$$
>
> When the reward (r) is less than 1, then increasing group size still reduces the probability of volunteering. However for larger rewards (when $r > 1$), this model results in a positive relationship between group size and volunteering.

Assuming a similar mistake to the previous paragraph, their Eq. 13 can be obtained by replacing Eq. 12 with

$$W_i = \gamma^{N-1}(1-a) + r^N(1-\gamma^{N-1}).$$

When $r > 1$ (e.g., $r = 1.1$), the probability that no one volunteers has a non-monotonic relationship with N for their parameter values.